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COMPUTATION OF REGIONS OF CONSTRAINED STABILITY FOR NONLINEAR CONTROL SYSTEMS

by Demosthenes P. Gelopulos and Donald G. Schultz

Prepared by
UNIVERSITY OF ARIZONA
Tucson, Ariz.

for

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ABSTRACT

The object of this study is to revise the classical control system stability criterion and to devise methods for determining the region of stability. The revision of the stability definition is motivated by the need for increased physical relevance. Excursion stability is defined in such a way that the practical constraints on the system are incorporated into the stability measure. Computational algorithms are devised which are used to determine the region of excursion stability. Two distinct algorithms evolve which are compared on the basis of computational efficiency.

The stability criterion is based on satisfaction of certain state-space constraints. The constraints are selected by considering the practical limitations of the system and the region of validity of the system model. The region of excursion stability consists of all those disturbed system states for which the ensuing trajectory does not violate the constraints.

The problem of representing the stability region is considered prior to the development of stability analysis techniques. The region of stability is approximated by an inscribed region of a prescribed shape. The prescribed shape is selected on the basis of the relative severity of the disturbances which are expected on each of the state variables.

The analysis problem is considered in the following context. Given a mathematical model of the control system in the form of state equations, and given also a set of state-space constraints, find the largest region of the given shape from which no emanating trajectories of the system violate the specified constraints. This analysis can only be performed on very simple systems unless a computational aid is used. Therefore, the analysis problem is reformulated into an optimization problem so that the computational methods already devised for the latter may be applied to the stability analysis problem.

The reformulated problem consists of a search for the reverse trajectory of the system equations initiating on the constraint boundary which minimizes a positive definite state function called a shape function. The reverse trajectory is a solution of the original system equations with time running backwards. One computational algorithm is generated by attacking the optimization problem with a random seeking method. Another algorithm evolves from the application of variational calculus to the optimization problem. The resulting two point boundary value problem is solved numerically using quasilinearization. Controlled experiments with both algorithms indicate that the random search is the better of the two methods.

Some suggestions are made for the development of synthesis procedures based on excursion stability.

There are essentially two contributions made in this work. First, the revision of the stability criterion ties stability analysis to the physical system and thereby makes stability analysis useful for engineering purposes. Another contribution of the work is the development of computational algorithms for evaluating the stability of nonlinear systems. A key feature of these algorithms is the fact that it is not necessary to select any state functions on any intuitive or semi-intuitive basis, as is the case with the second method of Liapunov. Therefore, the algorithms are not inherently limited to low-order systems.

Chapter One

Introduction

1.1 Introduction

Adequate methods have not yet been developed for the design of large dynamic systems. If the desired system operates in such a way that the physical quantities involved do not traverse a wide quantitative range, then some success can be enjoyed by linearizing the system under study and applying linear design techniques. Quite often the inherent nonlinearities of the system cannot be so ignored. Synthesis procedures applicable to systems which must be modeled by nonlinear describing equations are virtually nonexistent. The development of nonlinear design techniques must necessarily follow the establishment of meaningful goodness measures and system evaluation methods. The development of methods for the analysis of nonlinear systems based on a meaningful goodness criteria is the goal toward which the research described herein is directed.

The minimum requirements for usefulness of a system are referred to collectively as stability. Two questions immediately arise. What is meant, precisely, by minimum useability? How does one determine whether a given system is stable or not? There are several proposed stability criteria which are based on pleasing mathematical properties but suffer from a lack of engineering relevance. The problem of ascertaining the existence of stability has been considered in detail by many researchers with limited success.

The most basic requirements for usefulness of a system are the satisfaction of simple state-space constraints. Certain physical variables must be contained in some limited range if the system is to operate at all. Stability should be ascribed to those systems for which the excursion of the trajectories of the system from the possible disturbed states of the system satisfy the constraints. Excursion stability is difficult to establish by unaided computation. However, computer algorithms are discussed in a later chapter which allow the computation of regions of excursion stability on a computer.

1.2 Historical Background

The stability of nonlinear systems was first studied by the Russian Liapunov in 1892. Liapunov defined stability for a set of coupled first order differential equations by considering the properties of various autonomous solutions of the set of equations. Although he did not present a discursive algorithm for the determination of stability, he showed that a sufficient condition for stability was the existence of a state function with particular properties relevant to the system of equations.

In the early 1950's engineers became interested in Liapunov functions as a possible mechanism by which the stability of nonlinear control systems might be investigated. Ingwerson (1960), Schultz (1962) and others developed analytical techniques for generating Liapunov functions. The most recent technique was developed by Peczkowski (1966). The failure of any investigators to analyze high-order systems has inspired others to study the possibilities of using computational algorithms for the generation of Liapunov functions. Most of these studies are based on an observation by Zubov (1962), and all are attempts to determine stability by invoking the theorems based on Liapunov's method. Some well known studies of this type were conducted in this country by Margolis (1962), Rodden (1964), and Weissenburger (1965). It now appears that these computational methods are also limited to low-order systems.

The Rumanian mathematician, V. M. Popov, disclosed a method for demonstrating stability for a particular class of systems which is not limited to low-order systems, as in Aizerman and Gantmacher (1964). Higgins (1966) summarizes a number of extensions of Popov's theorem which broaden its applicability to a wider class of systems including some time-varying systems.

Popov techniques demonstrate asymptotic stability for a restricted class of globally stable systems. The power these methods have is evidenced by the susceptibility of high order systems to analysis. The most important limitation they have is that many practical systems of interest, such as systems with product-type nonlinearities, are not globally stable.

Much of the current work being conducted in stability analysis is concerned with extending the applicability of Popov-like methods. Very little attention has been directed at the evaluation of the engineering relevance of the existing stability criteria themselves.

1.3 The Need for a New Stability Criterion

Up to this time the stability analysis of nonlinear systems has met with limited acceptance by practicing engineers. This situation is due in part to the fact that methods for determining system stability for high-order systems are not readily available. Another reason for the inhibited use of stability analysis is the lack of correspondence between established stability and the satisfaction of minimum design requirements. In most applications the knowledge that a system is asymptotically stable does little to insure that the system will operate in an acceptable fashion.

What is needed is a stability criteria which considers the practical operational limits of the system. If a very slight or very probable disturbance can cause a system to operate in such a way that certain physical quantities, such as temperatures or velocities, will exceed their practical limits, then it is not comforting to know that the mathematical model representing the system will return to equilibrium from any disturbed condition.

1.4 Organization

Chapter Two contains a discussion of the classical stability criteria of importance and the analysis techniques based thereon. The motivation for this chapter is to briefly survey the most powerful methods available and to point out the difficulty associated with their application.

Excursion stability is defined in Chapter Three. The incentive for redefining stability is discussed in some detail. The analysis of systems based on excursion stability is shown to be related to an optimization problem. A number of examples is discussed which are intended to illuminate the peculiarities of excursion stability.

The methods by which the region of excursion stability are to be found are discussed in Chapter Four. These methods are in the form of computational algorithms, because the procedure requires the solution of nonlinear differential equations. There are two distinct types of algorithms discussed. One type involves the use of a seeking method. The other algorithm solves a nonlinear boundary-value problem.

Several examples are carried out in Chapter Five. It is shown by example that the algorithm which uses the seeking method is superior to the alternate algorithm in the sense that the computational requirements are greatly reduced.

The areas of study which are uncovered in this work are discussed in the concluding chapter. The logical extensions of this work fall into three distinct categories. It would be very convenient to be able to increase the efficiency of the algorithms enough so that a digital computer could handle large problems in less computing time. The algorithms presented require the availability of hybrid facilities to analyze complex systems. Now that the stability analysis in terms of excursion stability has been studied the door is open for the development of synthesis procedures based thereon. Some preliminary work relevant to both of these study areas is discussed in Chapter Six. Finally, it would be very desirable to make an extensive study of a large physical system based on the concepts presented here to test the asserted physical relevance of excursion stability.

1.5 Notation

A concerted effort is made to use the symbols most frequently associated with a particular variable in the existing literature. In some cases compromise is necessary to avoid symbolizing different variables redundantly.

Upper-case English characters are used to represent matrices and state functions. Column matrices or column vectors are denoted by underscores. Double character upper case letters are used to denote regions or sets of points in the state space.

Chapter Two

Asymptotic Stability

2.1 Introduction

Historically, the emphasis on asymptotic stability has been so overwhelming that the term is almost synonymous with stability. For this reason the propitious decision of some researchers to introduce the computer into stability investigations has not been accompanied by a concurrent reviewing of the classical stability criteria. In this chapter asymptotic stability is defined and the means of demonstrating its existence are summarized. The chapter considers asymptotic stability in enough detail so that it may be compared and contrasted with excursion stability, which is introduced in the next chapter.

The chapter is divided into three parts. The first part deals with the second method of Liapunov. The subsequent section contains a discussion of frequency-domain methods. Finally the numerical techniques for finding Liapunov functions are considered. The limiting disadvantages are noted in each case.

2.2 The Second Method of Liapunov

Suppose that a dynamic system is represented by

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t)) \tag{2-1}$$

where $\underset{\sim}{x}$ is an n by one matrix of time-dependent functions. The value of

$\underset{\sim}{x}$ at any particular time is called the state of the system at that time. The state of the system is often represented by a point in a Euclidian n-space known as the state space of the system. If $\underset{\sim}{x}(t)$ is a function which satisfies eq. (2-1) for all time, then the range of $\underset{\sim}{x}$ is called a trajectory of the system. Suppose the range of $\underset{\sim}{x}(t)$ is denoted by X. The point $\underset{\sim}{x}(t_1)$ is a member of X. Define the set of all points $\underset{\sim}{x}(t_2)$, where t_2 is greater than t_1, as the path from $\underset{\sim}{x}(t_1)$. Similarly, define the collection of points $\underset{\sim}{x}(t_3)$, where t_3 is less than t_1, as the historical path from $\underset{\sim}{x}(t_1)$. Define $\underset{\sim}{q}$ as an equilibrium point if

$$\underset{\sim}{f}(\underset{\sim}{q}) = \underset{\sim}{0}$$

Finally define RL(a) as the set of points in the state space bounded by the hypersphere of radius a.

Definition 2-1: The origin of the state space is stable in the sense of Liapunov if the origin is an equilibrium point of the system and if for any positive number, R, there exist another positive number, r, such that the paths from all points in RL(r) generate a subset of RL(R).

Definition 2-2: Suppose that the origin is stable in the sense of Liapunov. The origin is asymptotically stable if the Euclidian norm of all trajectories which initiate in RL(r) is bounded by an arbitrarily small positive number.

If it is possible to let r increase without bound as R increases, then the stability is called global. If the origin is asymptotically stable, but the stability is not global, then there are points about the origin from which the paths asymptotically approach the origin. The collection of all such points is called the region of asymptotic stability.

If the region of asymptotic stability has engineering relevance, then it is necessary to find some means of determining the region for a given system. The most obvious approach would be to determine all the trajectories of the system and classify those which approached the origin. However, solutions of nonlinear differential equations are rarely available in closed form. Liapunov suggests a second, or direct, method. If a state function can be found which exhibits certain properties relevant to the system under study, then it is possible to demonstrate stability without actually solving the system equations. There is a number of theorems available in the literature which state sufficient sets of conditions which Liapunov functions must satisfy. One of the most common of these theorems is reproduced here from Schultz, (1965) for convenience.

> Theorem 2-1: Suppose there exists a real scalar function $V(\underset{\sim}{x})$ defined, continuous with continuous first partials on $\underset{\sim}{x}$, such that $V(\underset{\sim}{x})$ is positive definite in a closed region U. One of the surfaces $V = K$ bounds U. The gradient of V on $\underset{\sim}{x}$, $\nabla_x V(\underset{\sim}{x})$ is not zero anywhere in U except at the origin. The time derivative of V along trajectories, $\overline{\nabla_x V(\underset{\sim}{x})^T \underset{\sim}{f}(\underset{\sim}{x})}$ is negative semidefinite in U and is not identically zero on a solution of the system other than the equilibrium point at the origin. Then the system is asymptotically stable in U.

The problem of demonstrating stability is transformed into a search for a state function which will satisfy the theorem. Liapunov has demonstrated, (1892), that if the system is stable in some region, then there exists at least one V-function which will show stability.

Several methods for finding V-functions appear in Gibson (1963), Schultz (1965), LaSalle and Lefschetz (1961), and Krassovskii (1963). A modification of the variable gradient method of Schultz has recently been published by Peczkowski (1966).

If a suitable V-function can be found, stability can be demonstrated for a whole class of systems. The effect of parameter variations on stability can be studied by analyzing the V-function. Another feature of the second method is that regions of stability may be found for systems which are not globally stable. The difficulties involved in finding regions of stability are profound, but the second method is at least theoretically applicable to systems which are not globally stable.

The practical significance of asymptotic stability is not revered by practitioners. Moreover, it is very difficult to find useful V-functions. A great deal of energy has been devoted to the generation of Liapunov functions for autonomous systems, and yet selected third-order systems are the most complex which are amenable to analysis. Time-varying systems present even more frustrating problems.

2.3 Absolute Stability

Absolute stability is globally asymptotic stability for the particular class of nonlinear systems described in the next paragraph. It was first studied by the Russian,Lure,who attacked the problem with the second method. The Lure procedure is discussed by Gibson, (1963), and the results of Lure are very difficult to apply. The most significant contribution to the study of absolute stability was made recently by the Rumanian, V. M. Popov. Popov used analysis to uncover a set of conditions in the frequency domain which demonstrate absolute stability as shown in Aizerman and Gantmacher (1964). Since Popov's work has been published, Kalman has unified the approach of Popov and Lure,as described by Higgins (1966). Others have extended the class of systems to which conditions similar to Popov's may be used.

A block diagram of the system studied by Popov is shown in Fig. 1. The block labeled G(s) represents a linear time-invariant system whose input-output transfer function is G(s). The nonlinearity $\phi(\sigma)$ is said to be in the sector zero-to-K if it is memoryless and satisfies

$$\phi(o) = 0$$

$$0 \leq \phi(\sigma) \leq K\sigma$$

The closed loop system is said to be absolutely stable in the sector zero to k if and only if the system is globally asymptotically stable for all nonlinearities, $\phi(\sigma)$ in that sector.

V. M. Popov demonstrated that the existence of a real number q, such that

$$\text{Real Part } \{(1 + j\omega q)G(j\omega)\} + \frac{1}{K} > 0$$

is satisfied, is sufficient to demonstrate absolute stability. The j in the inequality above is the complex number (0,1), and the inequality must be satisfied for all real non-negative values of ω. Satisfaction of the inequality above can be demonstrated graphically.

When the system under study falls into the class of systems which are amenable to analysis by the Popov method or its extensions, then stability can be shown rather easily even for high-order systems. If a given system is not absolutely stable it may still have a large region of asymptotic stability. For example, an adequate model of a nonlinear system often contains a product-type nonlinearity. Product nonlinearities may be represented by a set of four nonlinearities like

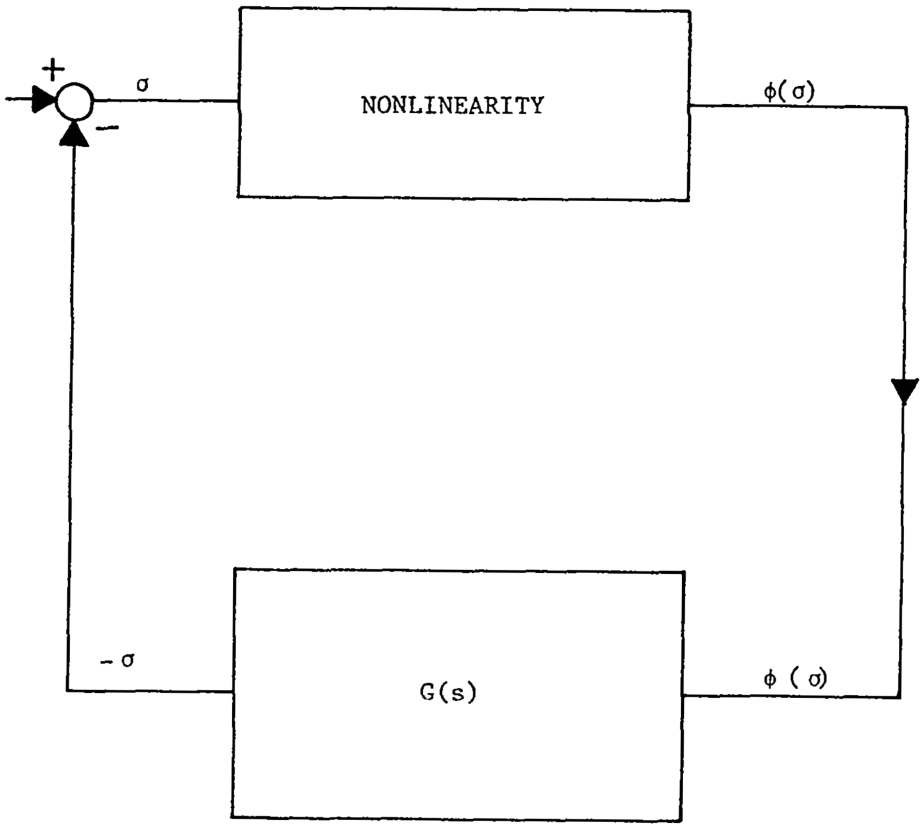

Figure 1. Block Diagram of System Studied by Popov

$$\phi_a(\sigma) = \frac{1}{8}\,\sigma(\sigma + |\sigma|)$$

and

$$\phi_b(\sigma) = \frac{1}{8}\,\sigma(-\sigma + |\sigma|)$$

If σ_1 and σ_2 are defined by

$$\sigma_1 = x_1 + x_2 \quad \text{and} \quad \sigma_2 = x_1 - x_2$$

then the product of x_1 and x_2 can be represented by

$$x_1 x_2 = \phi_a(\sigma_1) - \phi_b(\sigma_1) - \phi_a(\sigma_2) + \phi_b(\sigma_2)$$

Now the product looks like a device with four memoryless nonlinearities, each in the sector from zero to infinity. This configuration satisfies the requirements of one of the extended Popov methods described by Higgins, (1966).

The extension of the Popov theorem which applies is described below. Consider the system described by

$$\frac{d}{dt}\,\underset{\sim}{x}(t) = A\underset{\sim}{x}(t) - B\underset{\sim}{\Phi}(\underset{\sim}{\sigma})$$

where

$$\underset{\sim}{\sigma} = c^T\underset{\sim}{x}$$

$$0 \leq \phi_i(\sigma_i)/\sigma_i < \infty$$

and

$$\underset{\sim}{\Phi}^T(\underset{\sim}{\sigma}) = \lfloor \phi_1(\sigma_1) \quad \phi_2(\sigma_2) \quad \cdots \quad \phi_m(\sigma_m) \rfloor$$

The system is absolutely stable if there is a diagonal real matrix D such that

$$\mathrm{He}\Gamma = \mathrm{He}\{(I + j\omega D)c^T(j\omega I-A)^{-1}B\}$$

is a positive definite matrix, where He means Hermetian part. A second-order system which is linear except for one product-nonlinearity is described by the equations above if

$$B = \begin{bmatrix} b & -b & -b & b \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

The first and third element of Γ along the major diagonal are

$$\gamma_{11} = \frac{(1 + j\omega d_{11})(j\omega - a_{22} + a_{21})b}{a_{11}a_{22} - a_{12}a_{21} - j(a_{11} + a_{22})\omega - \omega^2}$$

and

$$\gamma_{33} = \frac{-(1 + j\omega d_{33})(j\omega - a_{22} + a_{21})b}{a_{11}a_{22} - a_{12}a_{21} - j(a_{11} + a_{22})\omega - \omega^2}$$

If these are evaluated at $\omega = 0$, then it is seen that

$$\gamma_{11}\Big|_{\omega=0} = -\gamma_{33}\Big|_{\omega=0}$$

No choice of D will cause He Γ to be positive definite so the test fails.

Either the test is weak, or else there are no second-order linear-plus-product systems which are globally stable. The latter is true. There is another equilibrium point at

$$x_1 = \frac{a_{11}a_{22} - a_{12}a_{21}}{-a_{21}b} \quad \text{and} \quad x_2 = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}b}$$

This system can never, therefore, be globally stable.

The point is simply that there are whole classes of simple systems which are not globally stable and therefore are not subject to analysis by the most powerful analytical methods available today for nonlinear systems.

2.4 Numerical Techniques

Since the task of determining the stability of a given control system is a difficult one, there is some justification for calling on the computer for help. Ultimately, the use of the computer should inspire a reformulation of the problem into a framework which would be more natural to a numerical method than to an analytical method. The first generation of algorithms for determining stability have been attempts to apply the second method directly by using the machine to generate Liapunov functions. Most of this work is based on the partial differential equation of Zubov.

Zubov, (1962) proposed a formal V-function which provides both necessary and sufficient conditions for stability. If two positive definite state functions, V and ϕ, exist such that

$$\overline{\nabla_x V(x)}^T \underset{\sim}{f}(\underset{\sim}{x}) = -\phi(\underset{\sim}{x}) \; (1 - V(\underset{\sim}{x})) \tag{2-2}$$

then the system is stable in the region bounded by the surface

$$V(x) = 1$$

and all the points outside of this region are unstable. Equation (2-2) is the relationship between V and the derivative of V along trajectories. It is known as Zubov's partial differential equation.

An algorithm for the solution of eq. (2-2) is discussed by Mragolis and Vogt (1963). Assume a polynomial form for $\phi(\underset{\sim}{x})$ and approximate $\underset{\sim}{f}$ with a polynomial. Establish a formal expression for V consisting of homogeneous polynomial forms with undetermined coefficients. Plug V, $\underset{\sim}{f}$, and ϕ into eq. (2-2) and retain all terms of order k + 1 or less, starting with k = 1. Equate coefficients of terms with like powers of $\underset{\sim}{x}$ components. This yields exactly the number of algebraic equations needed to solve for the unknown coefficients associated with the k = 1 in order terms in V. If the process converges, then it converges to a useful V-function for the system.

Unfortunately, the initial choice of ϕ is critical. If V is transformed inside the stability region by

$$u(\underset{\sim}{x}) = \ell n(1 - V(\underset{\sim}{x}))$$

then the derivative of the new Liapunov function, U, along trajectories is

$$\overline{\nabla_x U(\underset{\sim}{x})}^T \underset{\sim}{f}(\underset{\sim}{x}) = -\phi(\underset{\sim}{x})$$

Now it is clear that selection of ϕ is not unlike the selection of a

Liapunov function. Nevertheless, Hahn (1963), Margolis (1962), and Rodden (1964) have found V-functions for third-order systems using the Zubovian algorithm. However, if one hopes to apply the algorithm to higher order systems, then a basis for the selection of ϕ must be established.

2.5 Conclusion

At this time there is no generally applicable method for establishing the stability of a nonlinear system. The Popov method can be used on high-order systems, but its applicability is limited to globally stable systems. Regions of stability can be found if a suitable V-function can be generated. V-functions are hard to find, even with the help of a computer.

If the region of asymptotic stability is determined, it is still difficult to impart practical significance to the region.

This is the position of stability theory generally. Much attention is currently being directed at the broadening of the applicability of Popov methods. However, the lack of engineering significance suffered by asymptotic stability is receiving very little attention.

Chapter Three

EXCURSION STABILITY

3.1 Introduction

The shortcomings of asymptotic stability from a utilitarian viewpoint were mentioned in Chapters One and Two. In this chapter excursion stability is defined as an alternative to asymptotic stability, such that the satisfaction of a minimum design criterion implies excursion stability. The problem of representing the stability region is also considered, and a technique for approximating the region of stability is suggested. Finally, the properties of excursion stability are discussed prefatorily in anticipation of their use in Chapter Four.

3.2 Definition of Excursion Stability

There are so many definitions of stability for nonlinear systems that the definition of yet another requires some justification. Before the introduction of electronic computers, it was necessary to study nonlinear systems with pen and paper. The definitions of stability emphasized the ultimate behavior of the system trajectories in time because hand calculation is easier in the limit. The most undesirable character of the response of a given system to a disturbance, from an engineering viewpoint, is often exhibited soon after the occurrence of the disturbance.

It is very common to study the global properties of a system model. Attention directed toward the extremities of the state space may be expedient, but it is never necessary. The generation of the model is always accompanied by simplifying assumptions which are valid in a restricted region of the state-space. There is also a region in the state-space in which the physical system cannot operate without possible self-destruction, unreasonable expense, hazard or incompatibility with other systems.

A realistic appraisal of the system should indicate those initial states of the system from which the emanating trajectories will at some time violate the practical limitations of the system. Such a stability concept has greater engineering significance than any of the classical types of stability.

The following discussion illustrates how the limitations of a system may be incorporated into the stability concept. Consider the missile in Fig. 2. The missile is ideally traversing a vertical trajectory in a two-dimensional environment. Let x_1 denote the angular deflection of the missile axis from vertical. Let u denote the angle of the thrust vector relative to the missile axis. Suppose the missile dynamics are represented formally by a second order equation

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),u)$$

Suppose, further, that a closed loop system is designed by setting u equal to a linear combination of state variables so that

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),c_1x_1 + c_2x_2)$$

There is a limit on the magnitude of the attitude error because the

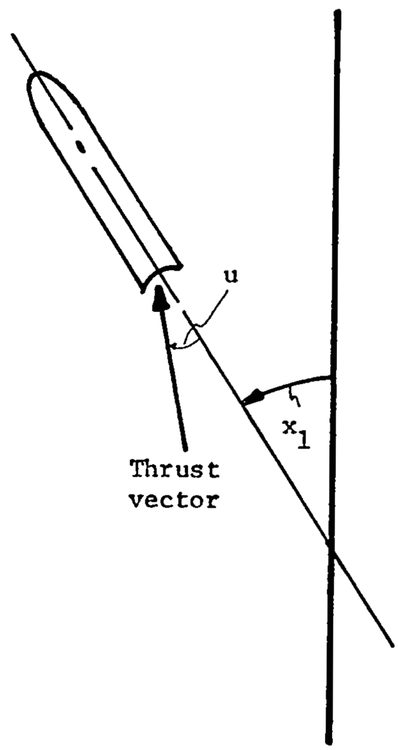

Figure 2. Missile in Example of Section 3.2

operation of other systems depends upon adequate performance of the attitude control system. This limitation is considered by enforcing an inequality constraint on the magnitude of x_1 such as

$$|x_1| \leq |x_1|_{max}$$

The constraint on x_1 is indicated by the pair of vertical lines on either side of the x_2 axis on the state-plane diagram shown in Fig. 3. There are also limits on u. It is not feasible to control the thrust vector over a range of more than a few degrees. If the limit on u is exceeded, the model no longer describes the system. Since u is a combination of state variables, the control law may be used to express the limitations of u in terms of a state space constraint

$$|u| = |c_1x_1 + c_2x_2| \leq |u_{max}|$$

The boundaries related to these limitations are shown in Fig. 3. When all the constraints are considered together, the region of the state plane in which the operation of the system must be confined is seen to be the set of all points bounded by the parallelogram in Fig. 3. Those trajectories which traverse this boundary indicate failure of the attitude control system, regardless of whether or not these trajectories are ultimately approaching the desired equilibrium point. That is, failure is indicated regardless of whether or not the describing equations are asymptotically stable.

<u>Definition 3-1</u>: Suppose that the operation of a system described by

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),t) \quad t \geq t_o \tag{3-1}$$

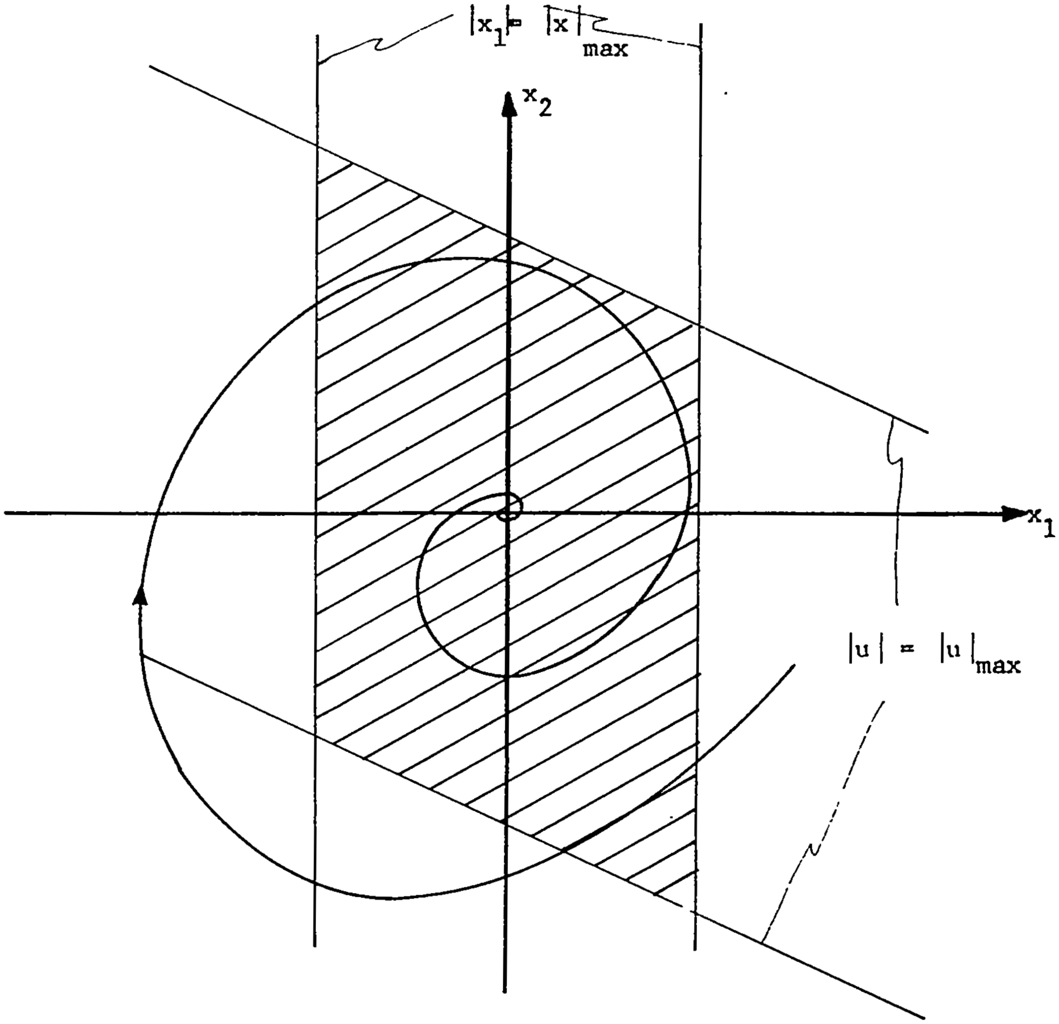

Figure 3. Missile Phase-Plane for Example of Section 3.2

must be limited to a region of the state space designated by RC. Denote the region of excursion stability by RS. A point p in the state space is a member of RS if and only if there is no point on the path from p which is not in RC. That is, if $x(t)$ satisfies eq. (3-1), then $x(t_1)$ is in RS if and only if t_2 greater than t_1 implies that $x(t_2)$ is in RC.

The following geometrical interpretation is possible for low-order systems. The boundary of RC encloses the excursion of all trajectories which initiate in RS. Figure 4 shows some selected trajectory segments on the state-plane of a second-order system. The RC boundary for this example is given by the vertical line shown. The points p_1, p_2 and p_3 are not in RS since the emanating paths violate the constraints. Point p_4 is not even a candidate for a member of RS because it is not in RC. The points p_5, p_6, p_7 and p_8 are all in RS. The state plane is redrawn in Fig. 5 with the entire region RS shaded.

Theorem 3-1: No trajectories emanate from RS.

Suppose p_1 is any point in RS and p_2 is any point outside of RS. By definition it is possible to go from p_2 to the complement of RC and it is not possible to go from p_1 to the complement of RC. Therefore, a contradiction arises if it is assumed possible to get from p_1 to p_2.

The behavior inside RS is not specified by definition 3-1. It is quite possible that the behavior of the system inside RS is not at all satisfactory. Consider a second-order system with two concentric limit cycles as shown in Fig. 6. The inner limit cycle is stable and the outer limit cycle is unstable. The region RC is bounded by some line in the state plane which is wholly outside of the outer limit cycle. The outer limit cycle bounds RS. Inside RS there is an unstable equilibrium point

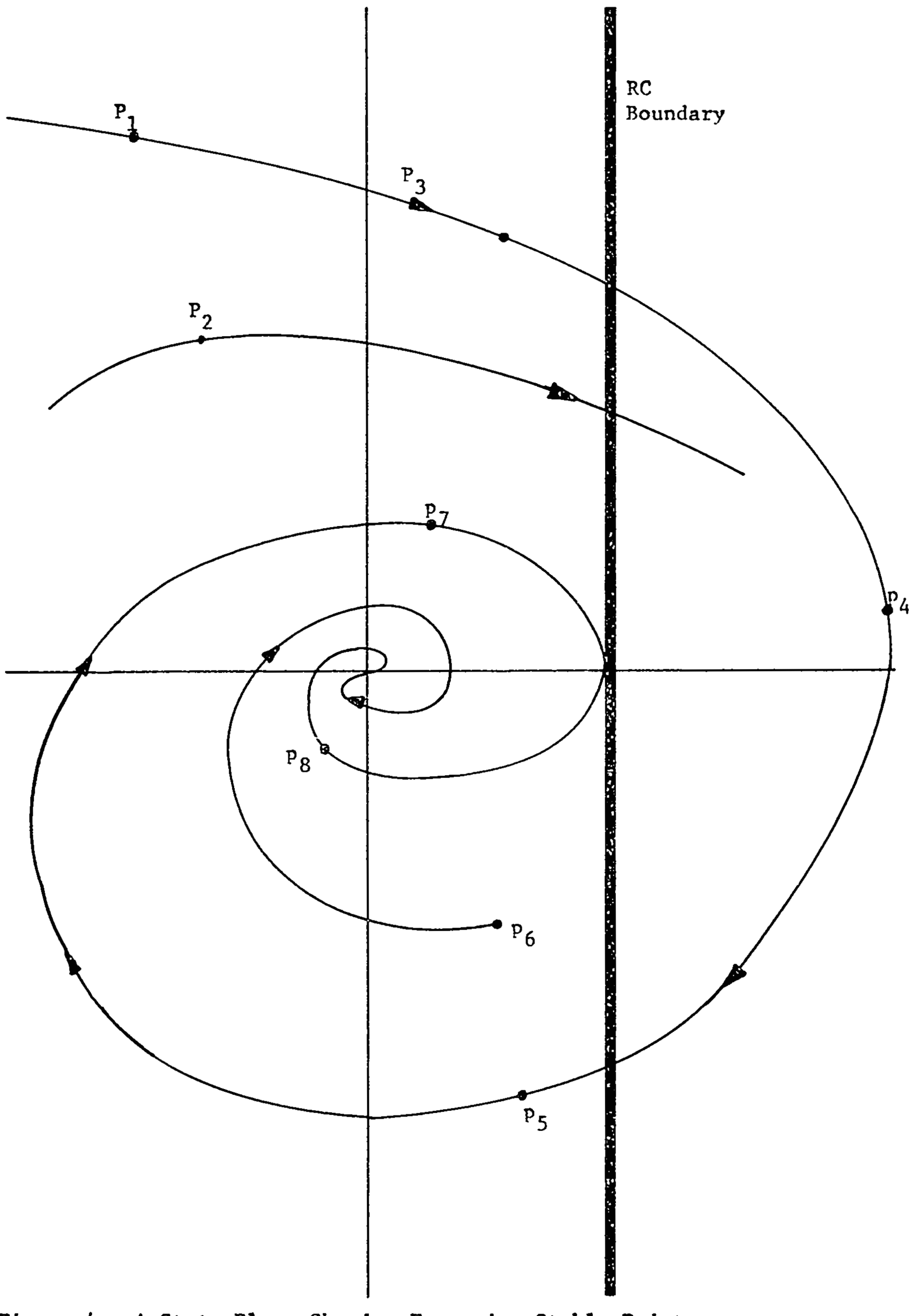

Figure 4. A State-Plane Showing Excursion Stable Points

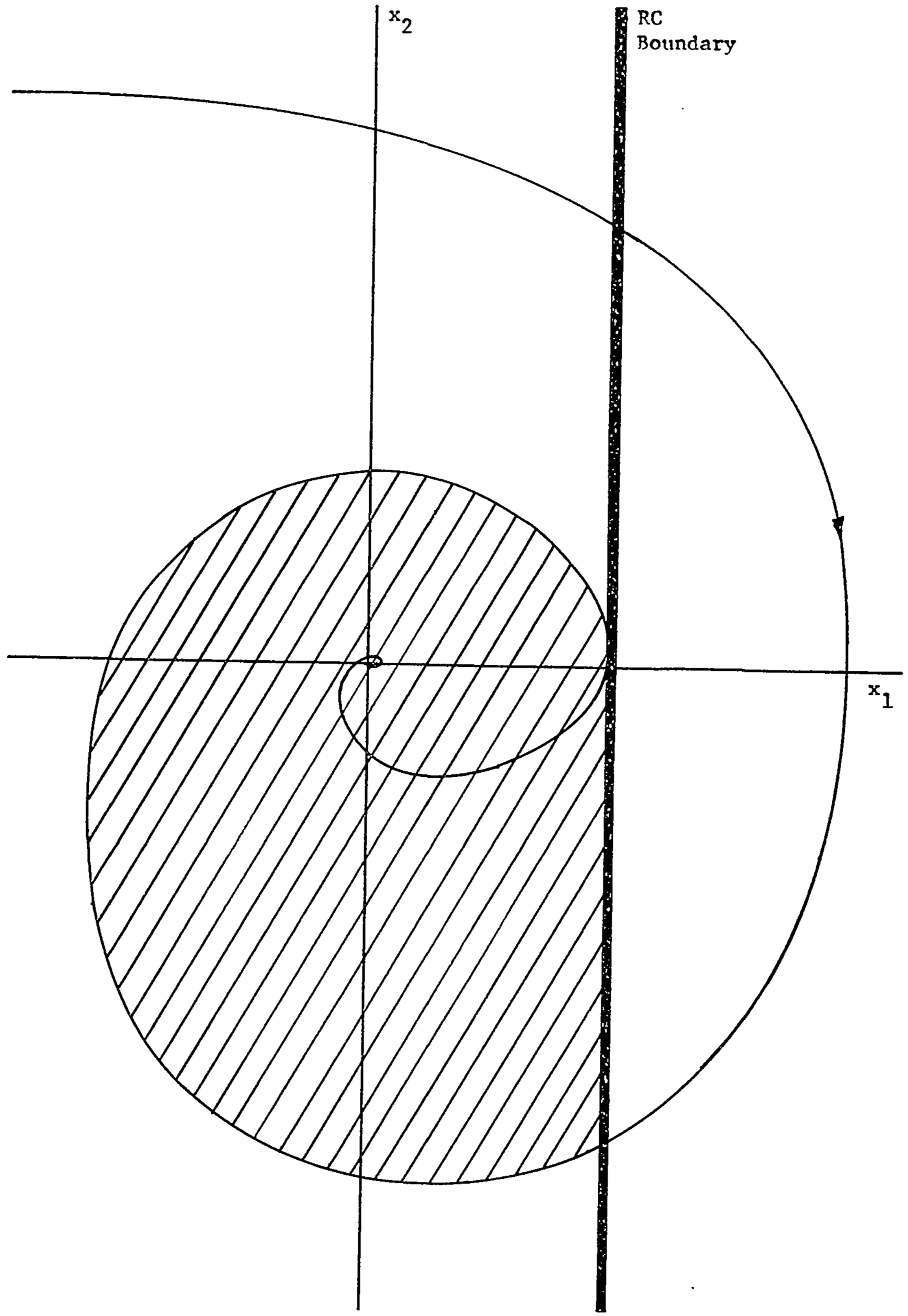

Figure 5. A State-Plane Showing a Region of Excursion Stability

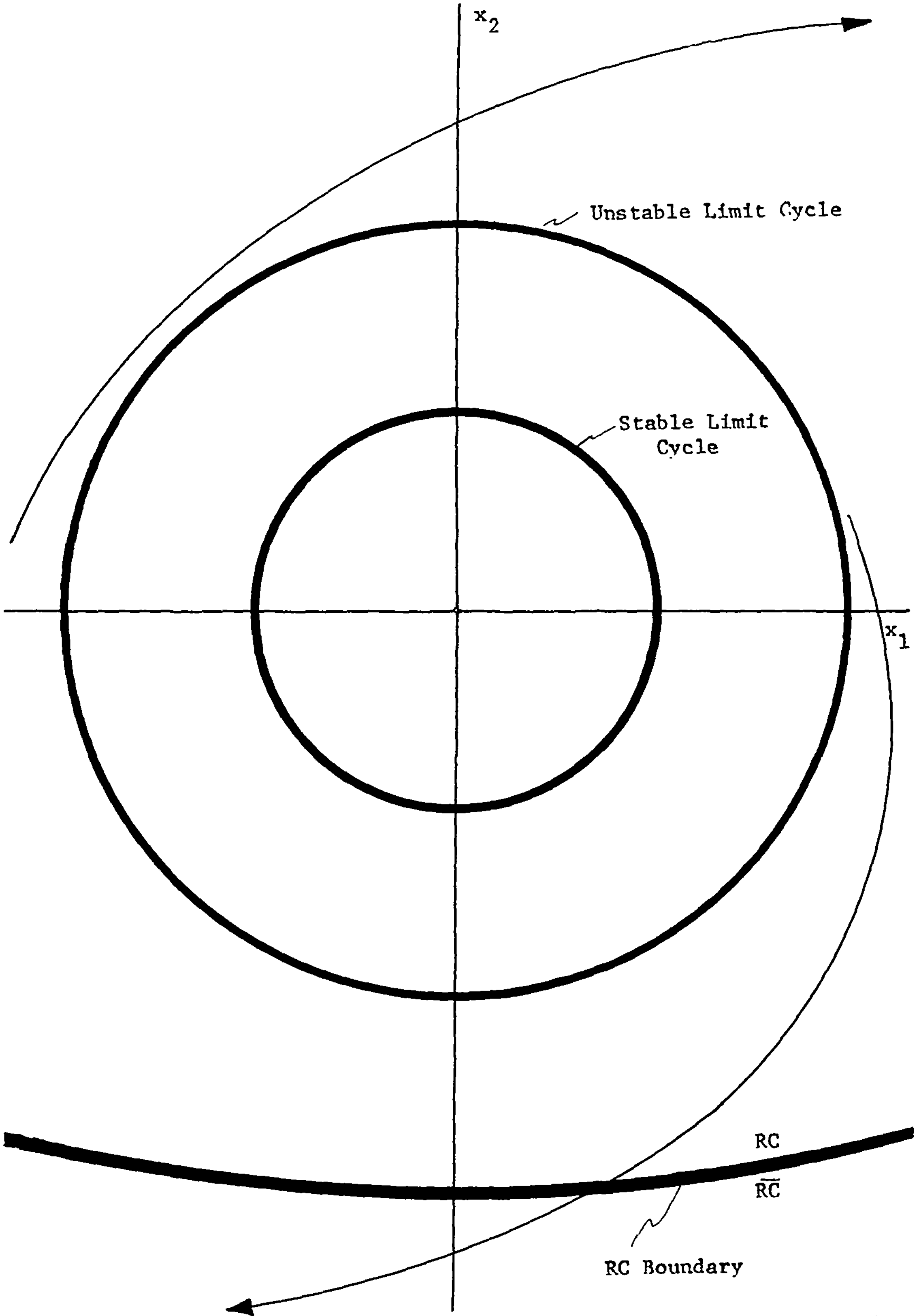

Figure 6. An Excursion Stable System which is Asymptotically Unstable

at the origin and the stable limit cycle. All trajectories initiating in RS are attracted to the limit cycle.

For most engineering problems it is desired that the system operate in such a way that a designated equilibrium point attracts all trajectories which are likely to occur. Therefore, the definition of excursion stability is modified in section 3.10 to preclude the existence of limit cycles or attracting surfaces inside RS.

3.3 The Representation Problem

The problem of adequately representing the region of stability has received little attention in the literature. A good method of representing the region of stability can be chosen only after one ascertains what information contained in knowledge of the stability region has relevance to the engineer.

If the second method of Liapunov is used to demonstrate stability for a nonlinear system, then the V-function is used to specify the region of asymptotic stability. The region is composed of the set

$$\{\underset{\sim}{x} \mid V(\underset{\sim}{x}) \leq K\}$$

The region is interpreted geometrically for low order systems. George (1966) developed a method for finding the largest right rectangular hyperparallelepiped which can be inscribed in the region described above when K is specified. Generally, however, there is not a great deal of incentive for studying the interpretation of regions bounded by V functions, since there is no way of finding V-functions for high-order systems.

The region of excursion stability demonstrates the inadequacy of a regulator system if it shows that there are disturbed states "near" the desired equilibrium point from which the emanating trajectory violates the system constraints. The meaning of nearness is designated on the basis of the physical situation. For example, if a problem has been scaled so that the system is likely to suffer disturbances of any one state variable which are the same order of magnitude as the disturbances which are likely to be imposed on any other state variable, then the Euclidian norm is a reasonable way to define nearness to the origin. In such a case the knowledge of the largest inscribable hypersphere in RS may be as valuable as a detailed description of RS, for purposes of system evaluation.

It is convenient to define a state function called a shape function, so that a suitable approximation to RS can be found. The shape function is a positive definite state function defined on the state space and selected on the basis of what is considered a good measure of nearness. Designate the shape function by S. Define a region bounded by a locus of constant S as the closed region RA(S,N) as given by

$$RA(S,N) = \{\underset{\sim}{x} \mid S(x) \leq N\}$$

Now it is possible to define an approximate region of excursion stability designated RS(S).

<u>Definition 3-2</u>: Suppose that a system has a region of excursion stability denoted by RS. Suppose, further, that a shape function S has been selected. Then <u>RS(S)</u> is defined as

$$RS(S) = RA(S,\bar{N})$$

where $\bar{N}$ is the least upper bound of all N for which

$$RA(S,N) \subset RS \tag{3-2}$$

is satisfied.

Figure 7 shows several loci of constant S on a state plane. The region RS is shown in the same figure. The region RS(S) is shaded. RS(S) is simply the largest region, whose shape is designated by S, which may be inscribed in the region RS.

3.4 Reformulation of the Stability Problem

The analysis problem is to find RS(S) for a given system, after RC and S have been chosen. The determination of RS(S) is not accomplished until Chapter Four. In this section the analysis problem is transformed into a variational problem.

Notice that in Fig. 7 there is a point on the boundary of both RS and RS(S). Such points are called critical points. There is always at least one critical point. If a region RA(S,N_1) has no boundary points in common with the boundary of RS, then RA(S,N_1) cannot be RS(S) because there must be a number N_2 greater than N_1 such that RA(S,N_2) is still contained in RS, as in Fig. 8.

Another very important observation may be made from Fig. 7. If RS is an open region, then the minimum value of S in the complement of RS occurs at a critical point. If RS is a closed region, then the boundary of RS does not belong to the complement of RS, but the greatest lower bound of all evaluations of S in the RS complement still occurs at a critical point. This simple observation is the key to the techniques for the determination of RS(S).

Theorem 3-2: The greatest lower bound for all values of the shape function in the complement of RS occurs at a critical point.

Let $\hat{\underset{\sim}{x}}$ be any point in the complement of RS. Since RS(S) is a subset of RS, $\hat{\underset{\sim}{x}}$ must also be in the complement of RS(S). RS(S) is, by

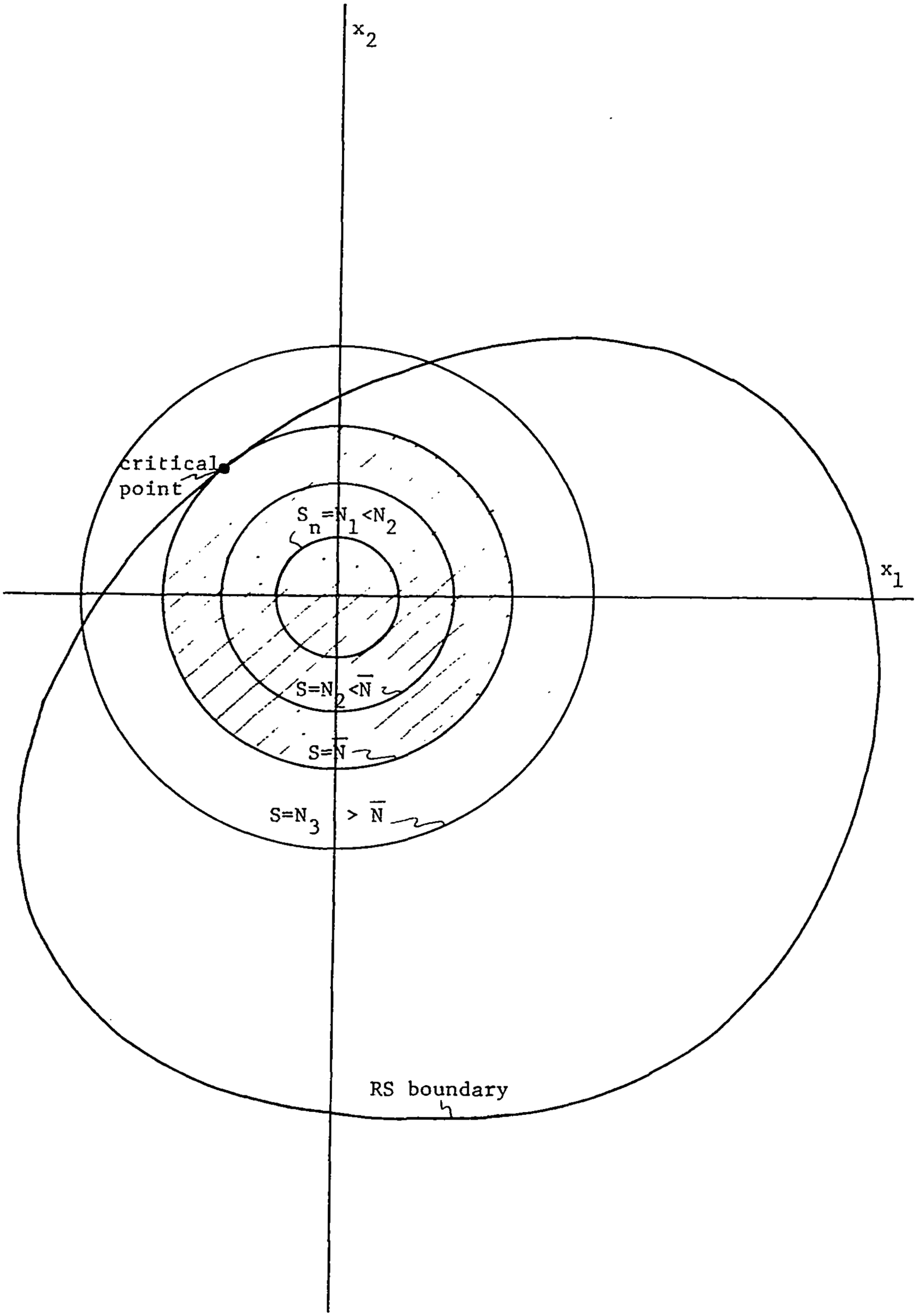

Figure 7. Some Loci of Constant Shape Function

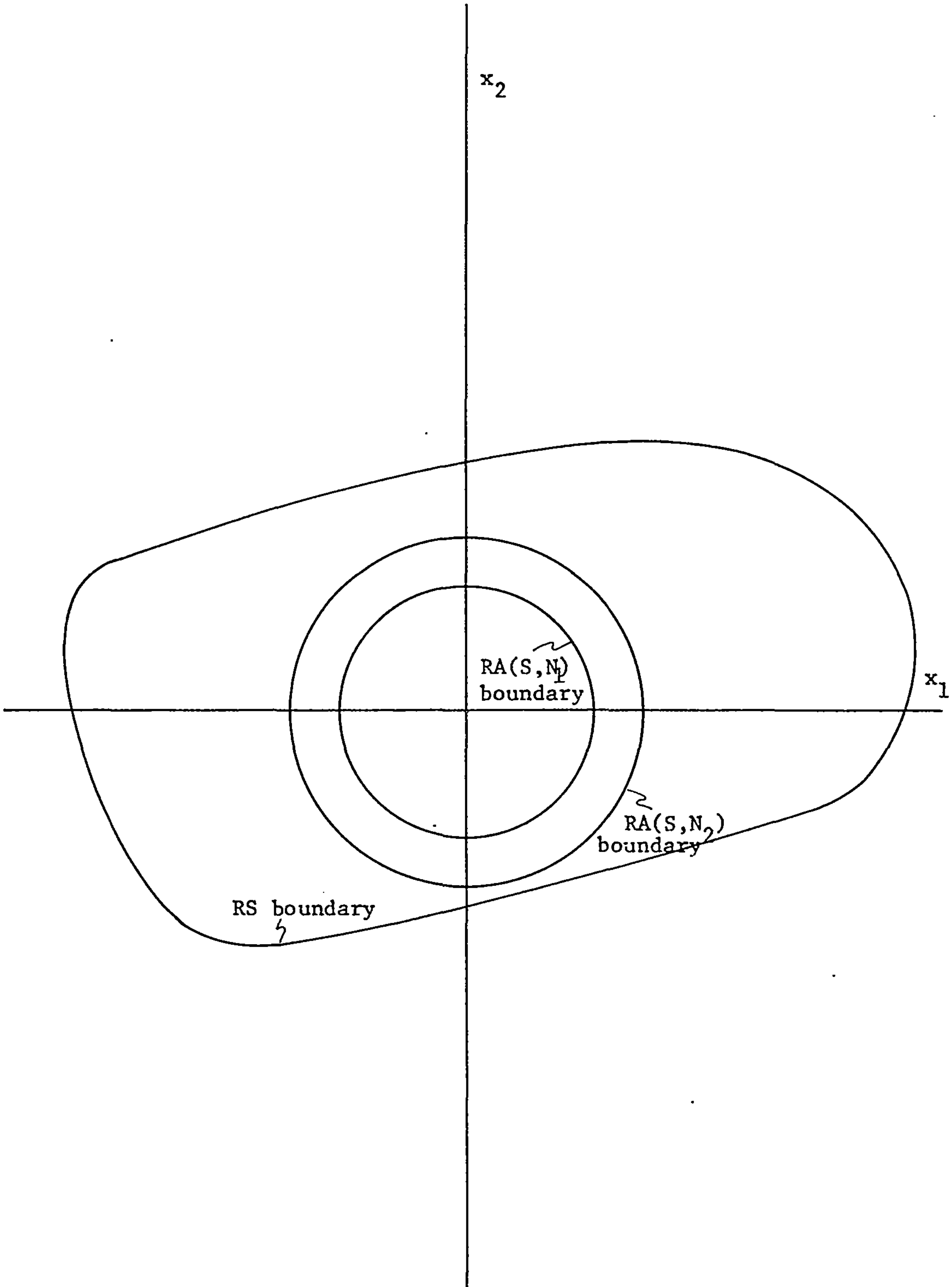

Figure 8. A Proper Subset of an Approximate Stability Region

definition, the set of all points for which the shape function is less than or equal to $\overline{N}$. Therefore, it follows that $\overline{N}$ is less than $S(\hat{x})$. Critical points fall on the boundary of RS(S). Therefore, S(x) is equal to $\overline{N}$ at a critical point. Consequently

$$S(\underset{\sim}{x}_{critical}) < S(\hat{\underset{\sim}{x}}) \qquad (3\text{-}3)$$

Suppose N is greater than $\overline{N}$. Since $\overline{N}$ is an upper bound for all N which satisfy relation (3-2), there must be a point $\hat{\underset{\sim}{x}}$, in the complement of RS at which the shape function has a value N. For any number larger than the value of the shape function at $\hat{\underset{\sim}{x}}$ there is a point in RS complement at which S has such a value. The value of S at any point in the RS complement is greater than the value of S at a critical point. The conclusion is that the value of S at a critical point is the greatest lower bound of all values of S in RS complement.

Notice that the critical points occur in RC or else on the RC boundary. This must be true since RS is a subset of RC. Therefore, the greatest lower bound of S in the complement of RS must occur inside or on the boundary of RC. It is sufficient to determine the complement of RS in RC. Refer to Fig. 4. Since it is possible to get from any point in the RS complement in RC to the boundary of RC along trajectories of the system, then by going backward in time along system trajectories, starting on the boundary of RC, it is possible to get back to any point in the complement of RS in RC. That is, the complement of RS in RC is generated by the historical paths from the RC boundary.

By combining the observation above with the theorem, the stability analysis is transformed into the following variational problem.

A critical point may be found by minimizing the shape function along reverse trajectories of the system which initiate on the RC boundary. The term minimize is used very loosely here. What is actually needed is a greatest lower bound. The problem of obtaining a greatest lower bound by use of techniques ordinarily intended for seeking minima is discussed in chapter four.

3.5 Maximally Stable Systems

If goodness is inferred by asymptotic stability, then it is possible that the region of asymptotic stability will fill the state space. Analogously, if excursion stability is synonymous with goodness, then the best one can expect is that RS fills RC. Systems for which RS and RC are identical are called maximally stable systems.

It is convenient, whenever possible, to identify the boundary of RC by an algebraic equation of the form

$$G(\underset{\sim}{x}) = 0$$

It is also desirable to select the state function G so that the following is satisfied

$$\underset{\sim}{x} \varepsilon RC \Leftrightarrow G(\underset{\sim}{x}) < 0$$

The constraint region defined below is used in many of the following examples

$$RC = \{\underset{\sim}{x} \mid x_1 \leq 1\} \qquad (3\text{-}4)$$

This is a simple region corresponding to an inequality constraint on one state variable. It is also a form of RC which is very likely to evolve in a practical problem. The G function for such a region is given by

$$G(\underset{\sim}{x}) = x_1 - 1$$

If x_1 is less than one, G is negative. Along x_1 equals one, G is zero. For x_1 greater than one, G is positive.

If RS and RC are identical, then there is no point in the interior of RC from which the emanating path reaches the RC boundary. Therefore, any trajectories of the system which intersect the boundary of RC must do so in such a way as to enter RC. The value of G along trajectories entering RC must be decreasing, because of the way that G was defined. Therefore, a sufficient condition for a system to be maximally stable is for the following inequality to be satisfied

$$\{\overline{\nabla_x G(\underset{\sim}{x})}^T \underset{\sim}{f}(\underset{\sim}{x})\} \Big|_{G(\underset{\sim}{x}) = 0} < 0 \tag{3-5}$$

The quantity in brackets is the time rate of change of G along trajectories. If the trajectories are entering RC at the RC boundary, then they are going from an area where G is positive to an area where G is negative. Therefore, G must be decreasing at all points on the RC boundary.

The use of condition (3-5) is illustrated by examples 3-1 and 3-2.

Example 3-1: Consider the system described by

$$\frac{d}{dt}\underset{\sim}{x}(t) = A\underset{\sim}{x}(t) = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix} \underset{\sim}{x}(t)$$

where RC is defined by eq. (3-4). An evaluation of the left side of condition (3-5) yields

$$-x_1(t) \Big|_{x_1 - 1 = 0} = -1 < 0$$

Condition (3-5) is satisfied. The state plane for the system is shown in Fig. 9. The trajectories which cross the line $x_1 = 1$ all cross from right to left into RC. No matter where the system starts in RC it cannot get outside of RC.

Example 3-2: A nonlinear example of a maximally stable system is given by

$$\frac{d}{dt}\,\underset{\sim}{x}(t) = \begin{bmatrix} -x_2^2(t) & 0 \\ \alpha(\underset{\sim}{x}) & \beta(\underset{\sim}{x}) \end{bmatrix} \underset{\sim}{x}$$

where RC is again defined by eq. (3-4). A possible state plane is shown in Fig. 10. The test for maximal stability, condition (3-5), is evaluated below

$$-x_2^2(t)x_1(t) \Big|_{x_1 - 1 = 0} = -x_2^2(t) < 0$$

The absence of maximal stability may be demonstrated by the antithesis of condition (3-5) as given below

$$\{\overline{\nabla_x G(\underset{\sim}{x})}^T \underset{\sim}{f}(\underset{\sim}{x})\} \Big|_{x = \tilde{x}} > 0$$

where $\tilde{x}$ is any point on the boundary of RC. It is possible that a given system does not satisfy either the condition above or condition (3-5). The state plane of a system which fails both conditions is shown in Fig. 11. The situation described above is surely a pathological case which one would not expect to encounter in practice.

Since RS and RC are the same for a maximally stable system, the largest region of shape S which is inscribable in RS is also the largest such region inscribable in RC. Designate the largest region of shape S which is inscribable in RC by $RA(S,N_{RC})$. For maximally stable systems

$$RS(S) = RA(S,N_{RC})$$

regardless of what positive definite shape function is chosen.

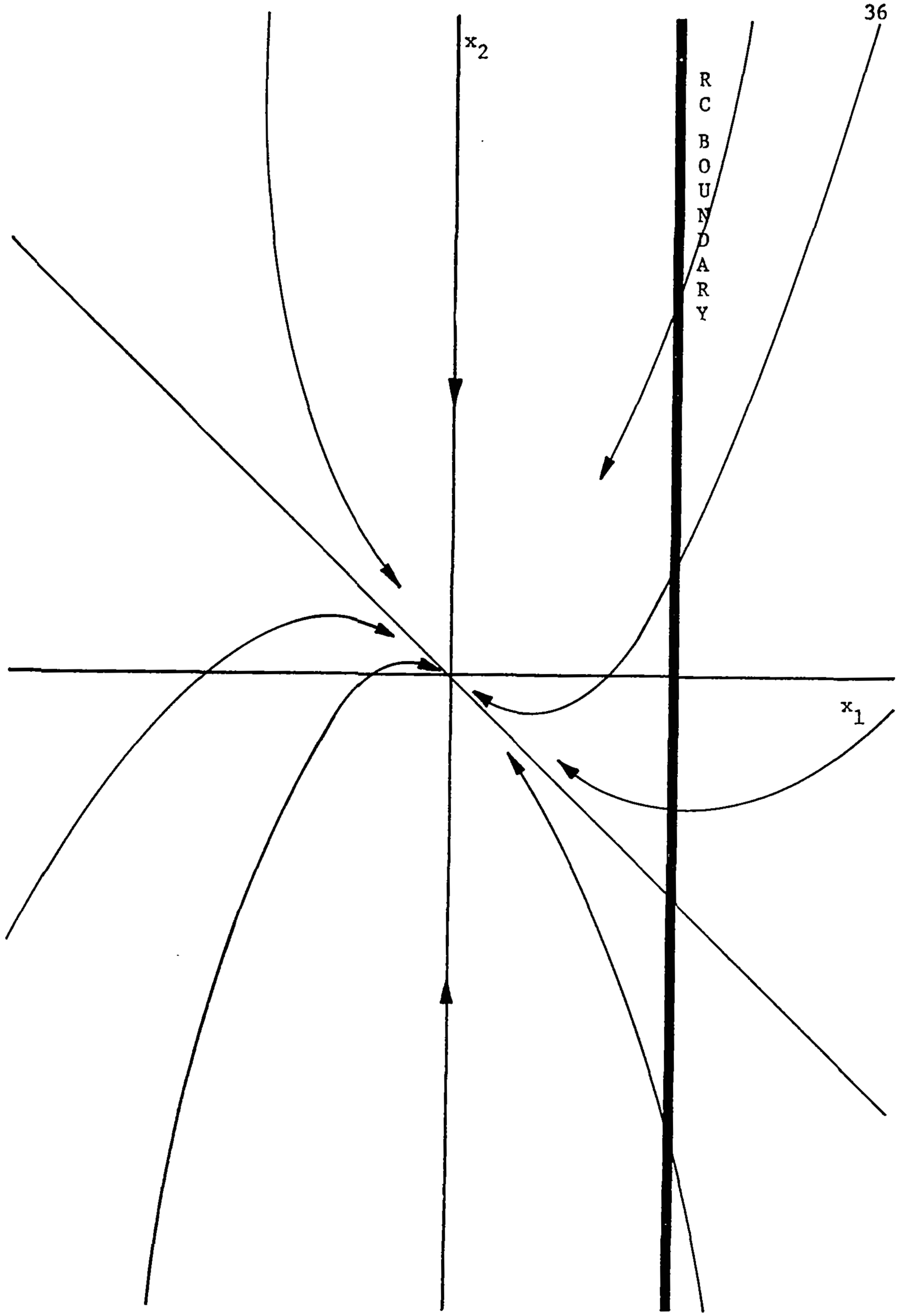

Figure 9. State-Plane for Example 3-1

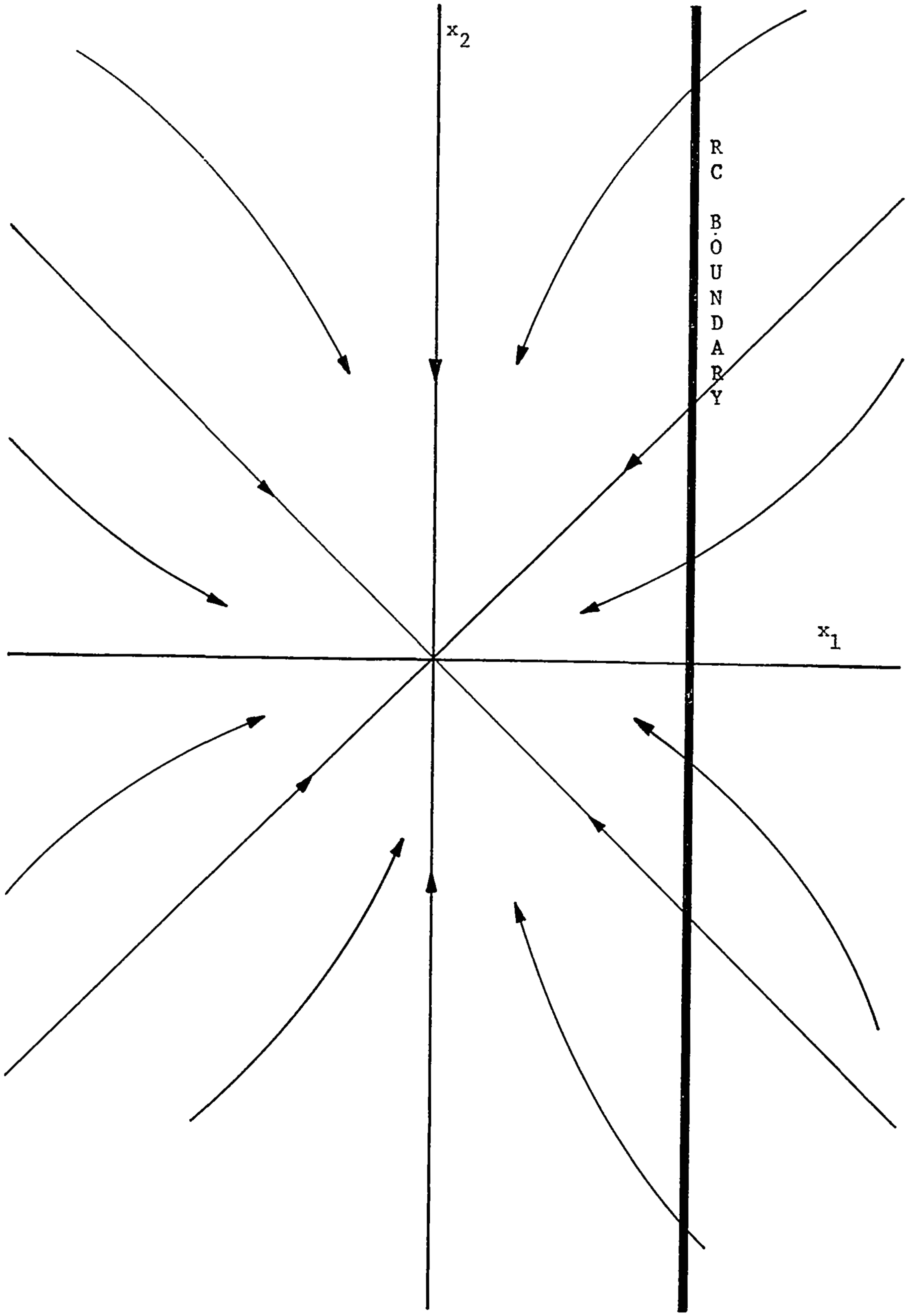

Figure 10. State-Plane for Example 3-2

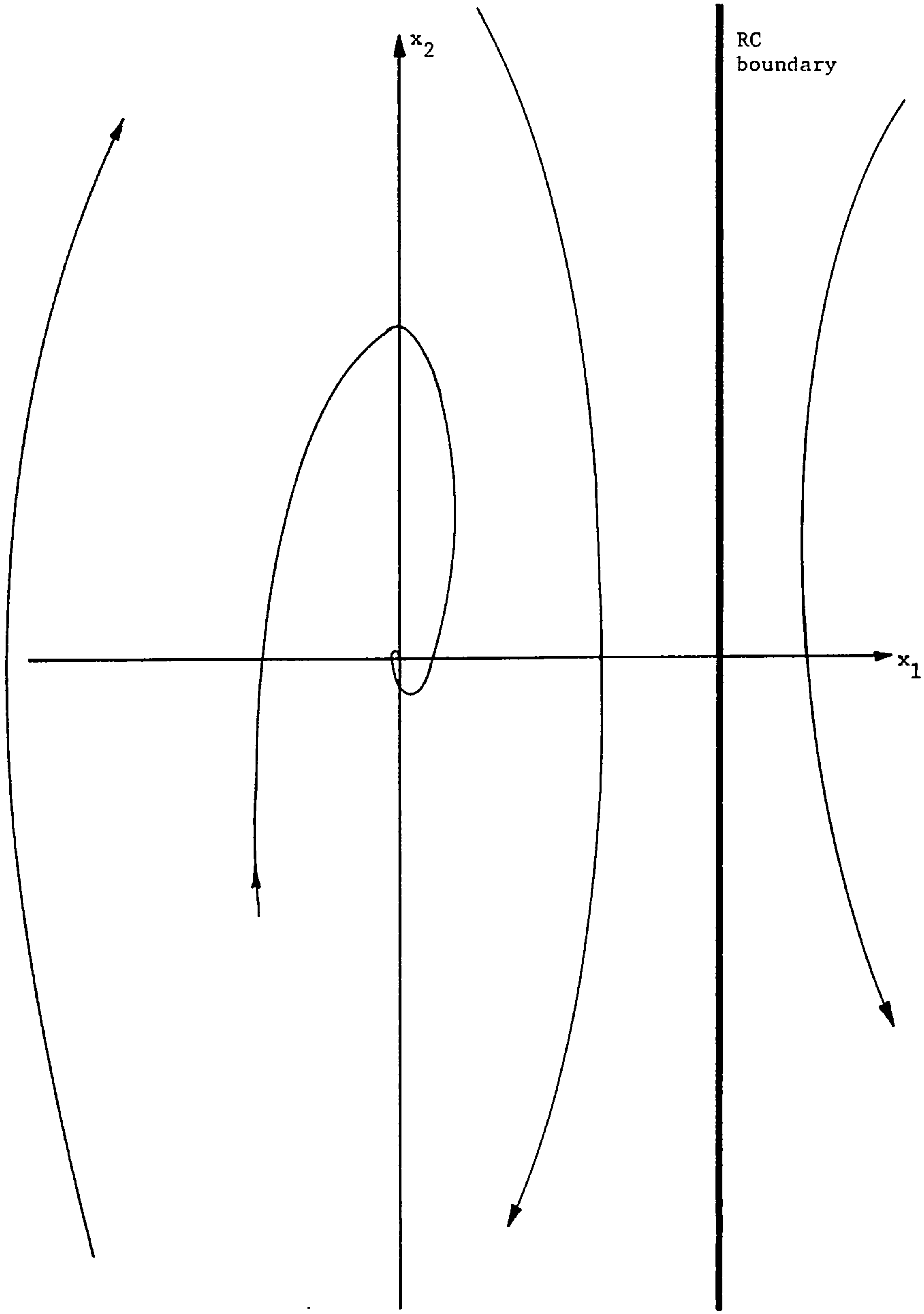

Figure 11. A System which Fails both Maximal Stability Tests

3.6 Approximately Maximal Stable Systems

The approximate region of stability, RS(S), cannot be any larger than $RA(S,N_{RC})$. If N is taken any larger than N_{RC}, then RA(S,N) includes some points which are not in RC. Define an approximately maximal stable system as a system for which RS(S) equals $RA(S,N_{RC})$. Approximate maximal stability is abbreviated AMS. All maximally stable systems are AMS. However, AMS does not imply maximal stability as shown by the counter example in Fig. 12. Clearly, a system may be AMS relative to one shape function and yet have a very small region RS(S) for another shape function.

There is no easy way to ascertain whether or not a given system is AMS. If all the trajectories which intersect the boundary of $RA(S,N_{RC})$ do so in an inward manner, then it is possible to demonstrate AMS by a condition analogous to eq. (3-5) above. The condition on trajectories intersecting a surface of constant S is

$$\{\overline{\nabla_x S(\underset{\sim}{x})}^T \underset{\sim}{f}(\underset{\sim}{x})\} \Big|_{S(\underset{\sim}{x}) = N_{RC}} < 0 \qquad (3\text{-}6)$$

This condition is sufficient to demonstrate AMS but it is not necessary. The system shown in Fig. 13 does not satisfy condition (3-6), but it is AMS.

The following example shows how condition (3-6) can be used to demonstrate AMS.

<u>Example 3-3</u>: Consider the system described by

$$\frac{d}{dt}\underset{\sim}{x}(t) = A\underset{\sim}{x}(t) = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \underset{\sim}{x}(t)$$

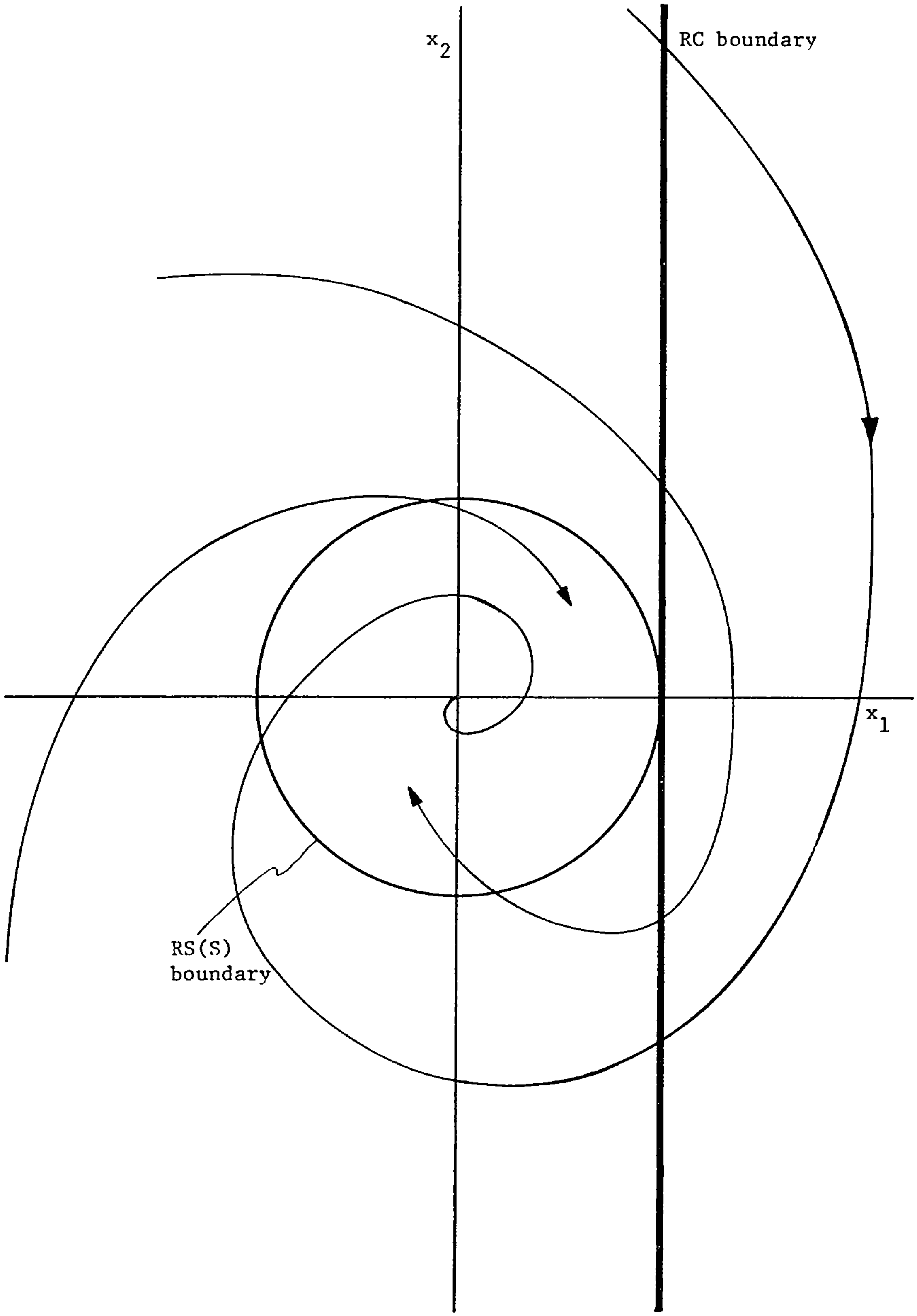

Figure 12. An AMS System which is not Maximally Stable

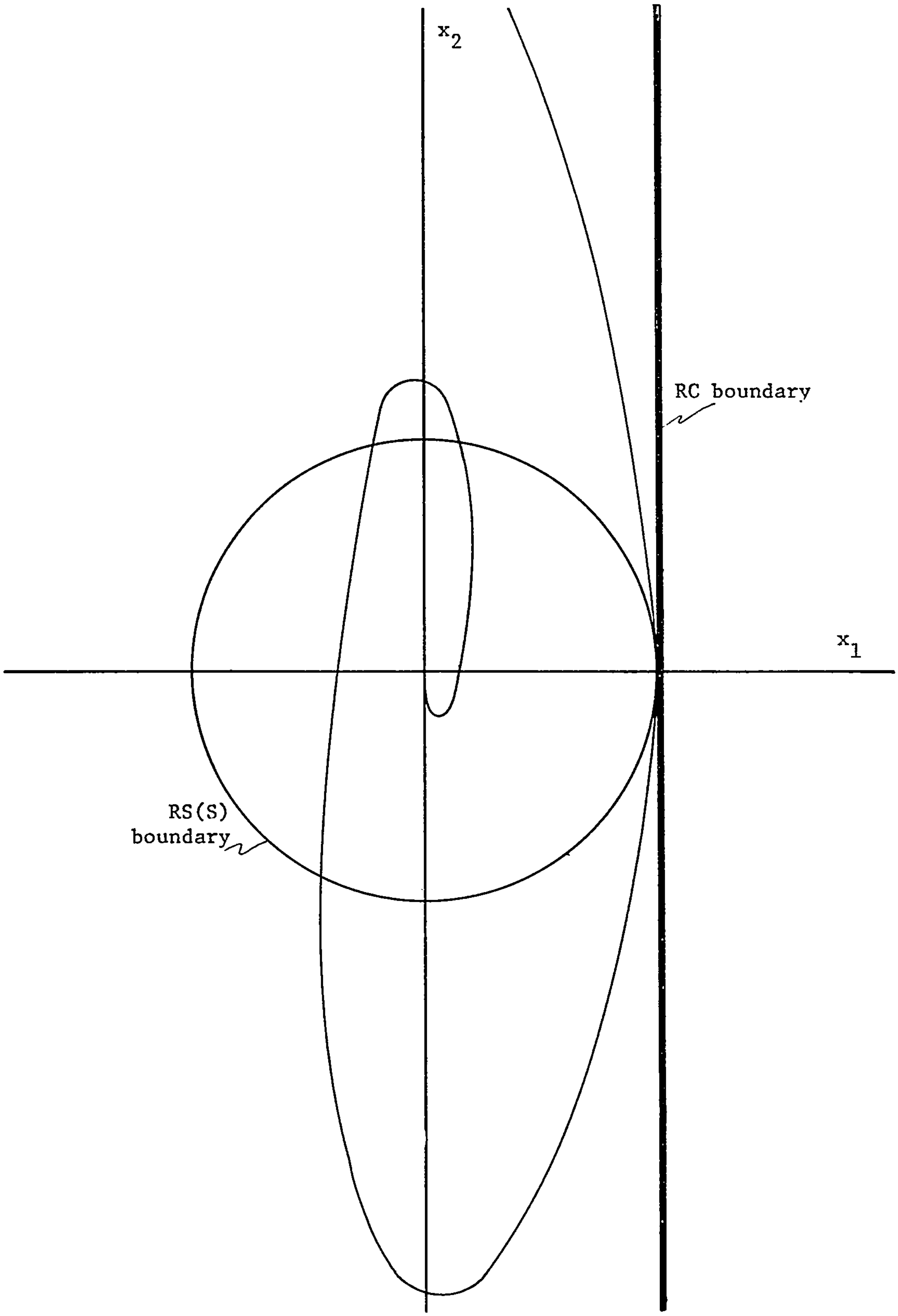

Figure 13. An AMS System which Fails Condition (3-6)

where RC is given by eq. (3-4) and S is the Euclidian norm. The largest circle which may be inscribed about the origin inside the region where x_1 is less than one is the unit circle. Therefore, N_{RC} must be unity. The state plane is shown in Fig. 14. Condition (3-6), as applied to this example, becomes

$$2\underset{\sim}{x}^T A \underset{\sim}{x} \Big|_{\underset{\sim}{x}^T \underset{\sim}{x} = 1} = -2 - 2x_2^2 < 0$$

Notice, from Fig. 14, that trajectories always enter the unit circle, but that they cross the RC boundary in both directions. The system has been shown to be AMS, but it is clearly not maximally stable. This observation is supported by evaluating the expression in condition (3-5) for the example as shown below

$$\left[\frac{d}{dt}(x_1 - 1)\right] \Big|_{x_1 - 1 = 0} = -1 + x_2 < 0$$

For all x_2 greater than one the condition is violated.

3.7 Systems which are not AMS

The analysis of systems which are not at least AMS generally requires the use of a computer. For selected simple systems an analysis can be made on paper. This analytical method applies only to a very narrow class of systems. However, the method is discussed in this section to preface the problem reformulation in Chapter Four.

Consider the system described by

$$\frac{d}{dt}\underset{\sim}{x}(t) = A\underset{\sim}{x}(t)$$

RC is again given by eq. (3-4). A is a square matrix of constants. The reverse system is given by

$$\frac{d}{dt}\underset{\sim}{y}(t) = -A\underset{\sim}{y}(t)$$

where $\underset{\sim}{y}(t)$ describes the historical trajectory. The initial condition

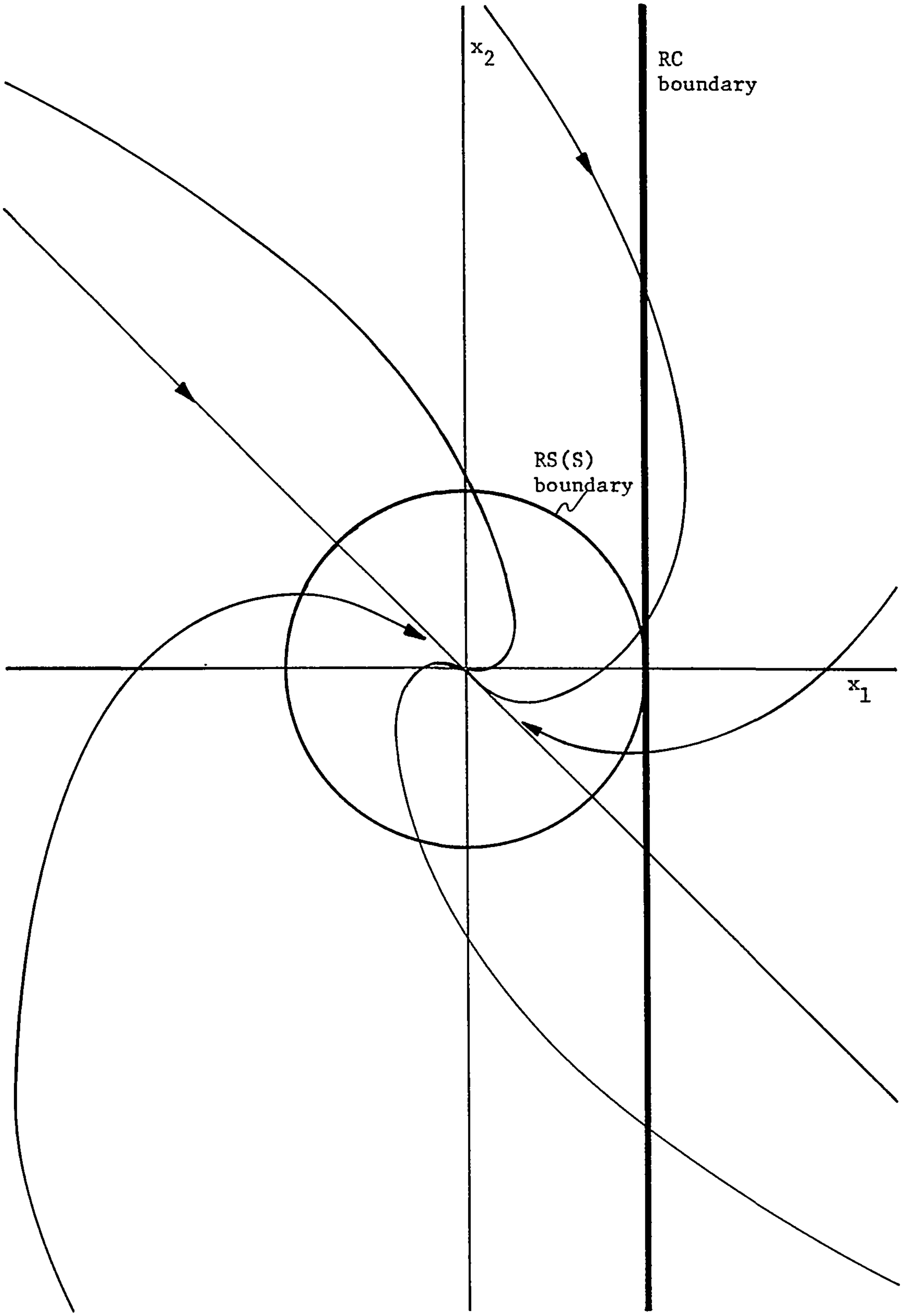

Figure 14. State-Plane of Example 3-3

imposed on the reverse system is

$$G(\underset{\sim}{y}(o)) = y_1(o) - 1 = 0$$

All the solutions to the reverse system equations subject to the initial condition above generate the complement of RS in RC, as discussed in section 3.4. The trajectories of the reverse system discussed above start on the line $y_1 = 1$. At least one of these trajectories traverses a critical point. Call the initial condition of this particular reverse trajectory $\underset{\sim}{z}$. Suppose that the reverse trajectory from $\underset{\sim}{z}$ reaches the critical point in time T. The situation being discussed is diagrammed for a second-order system in Fig. 15. The shaded area represents RS(S) and the trajectory shown is the one which intersects a critical point $\underset{\sim}{y}(T)$. The critical point can be related to $\underset{\sim}{z}$ by the transition matrix of the original system evaluated at -T as shown below

$$\underset{\sim}{y}(T) = \phi(-T)\underset{\sim}{z}$$

where the transition matrix is an n by n matrix of time functions which satisfies

$$\underset{\sim}{x}(t_2) = \phi(t_2 - t_1)\underset{\sim}{x}(t_1)$$

The critical point is also the point at which S is minimized along a reverse trajectory from the RC boundary

$$S(\underset{\sim}{y}(T)) = \underset{\substack{\tau > 0 \\ y_1(o)=1}}{\text{Min}} \{S(y(\tau))\}$$

Since the system is linear there is no possibility of a limit cycle. If the state variables have continuous time derivatives along trajectories,

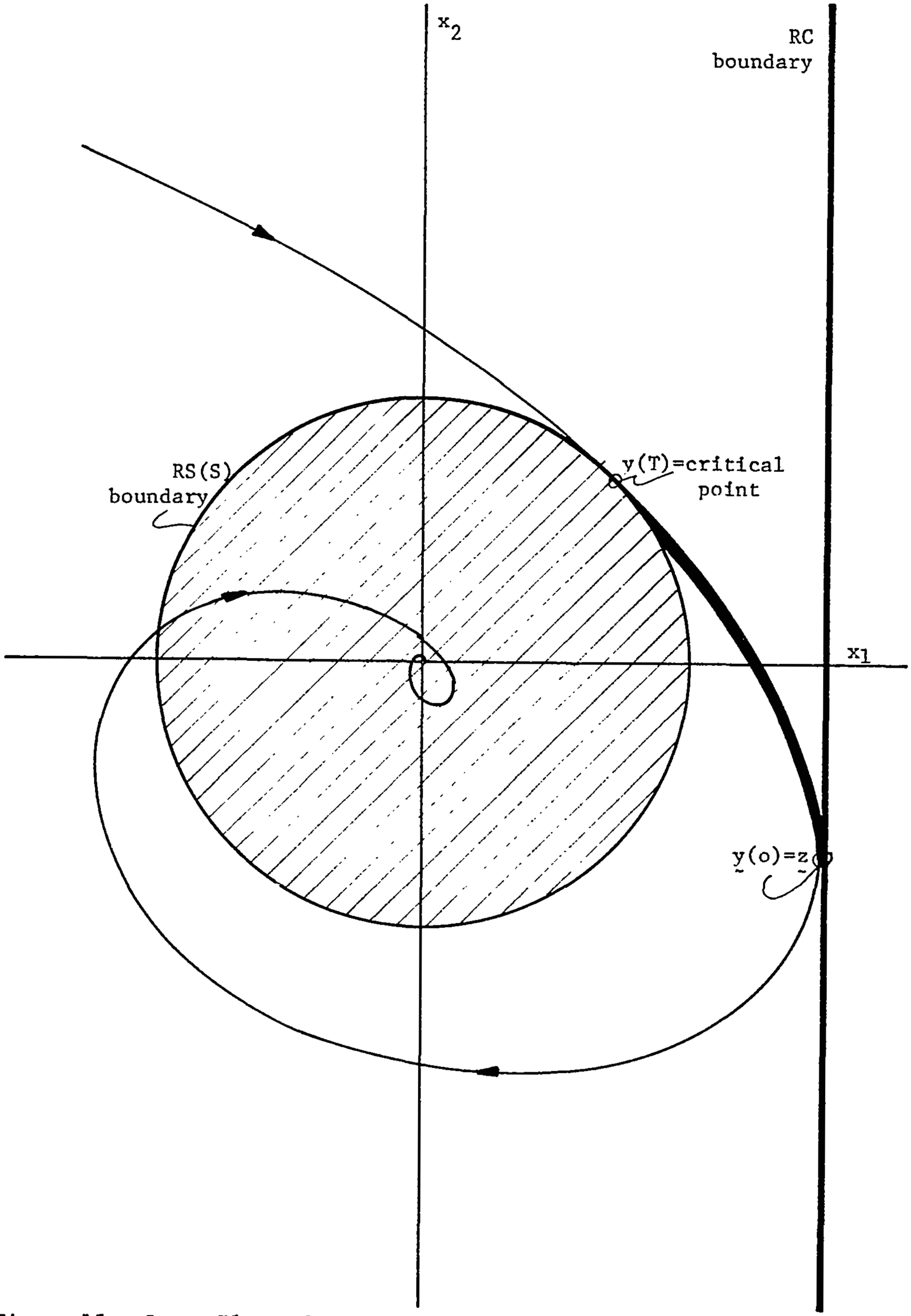

Figure 15. State-Plane Showing a Critical Reverse Trajectory

then the following conditions are necessary at the critical point

$$\partial/\partial z_K\{S(\phi(-T)\underset{\sim}{z})\} = 0 \qquad K = 2,3,\ldots,n$$

$$\partial/\partial T\ \{S(\phi(-T)\underset{\sim}{z})\} = 0 \tag{3-7}$$

These conditions represent a set of algebraic equations which have a critical point as a solution. Unfortunately, the equations are unbearably tedious to solve.

It is possible to devise a simpler method to solve second-order linear systems. Condition (3-7) can be put into a more convenient form by noting that the left hand side is merely the first time derivative of S along a system trajectory. This derivative of S may be expressed in terms of the system equations as

$$\{\overline{\nabla_x S(\underset{\sim}{x})}^T \underset{\sim}{f}(\underset{\sim}{x})\}\Big|_{\underset{\sim}{x} = \underset{\sim}{x}_{\text{CRITICAL}}} = \{\overline{\nabla_{\underset{\sim}{x}} S(\underset{\sim}{x})}^T A\underset{\sim}{x}\}\Big|_{\underset{\sim}{x} = \underset{\sim}{x}_{\text{CRITICAL}}} = 0 \tag{3-8}$$

Condition (3-8) has an interesting geometrical interpretation. The trajectory through a critical point is tangent to the surface of constant S at that point.

Consider only second-order problems with quadratic shape function given by

$$S(\underset{\sim}{x}) = \underset{\sim}{x}^T B\underset{\sim}{x}$$

where B is a constant symmetric matrix. For this case condition (3-8) reduces to

$$\{2\underset{\sim}{x}^T BA\underset{\sim}{x}\}\Big|_{\underset{\sim}{x} = \underset{\sim}{x}_{\text{CRITICAL}}} = 0 \tag{3-9}$$

This is a biquadratic equation in x_1 and x_2 such as

$$c_1x_1^2 + c_2x_1x_2 + c_3x_2^2 = 0$$

where c_1, c_2 and c_3 can be determined from A and B. The quadratic formula can be used to solve the equation above for x_1 in terms of x_2

$$x_1 = [-\frac{c_2}{2c_1} \pm \sqrt{(\frac{c_2}{2c_1})^2 - \frac{c_3}{c_1}}]x_2 = \frac{1}{c_{4,5}} x_2$$

This means that the critical points must lie along one of the two straight lines as given by

$$\underset{\sim}{y}(T) = \begin{bmatrix} 1 \\ c_{4,5} \end{bmatrix} b \tag{3-10}$$

where b is an unknown constant. The values of z_1 which can be attained from any of these points along system trajectories are

$$z_1 = b[\phi_{11}(T) + \phi_{12}(T)c_{4,5}] \tag{3-11}$$

If z_1 is considered to be free, then the maximum value of z_1 attainable from any of these points occurs when

$$b \frac{d}{dT} [\phi_{11}(T) + \phi_{12}(T)c_{4,5}] = 0$$

or equivalently when

$$\frac{d}{dT} \phi_{11}(T) = -c_{4,5} \frac{d}{dT} \phi_{12}(T) \tag{3-12}$$

From the equation above a number of solutions for T can be obtained. These candidates for T can be inserted into eq. (11). The right-hand side of eq. (3-11) is set equal to one and solved for the candidates for b. These b candidates can be inserted into eq. (3-10) to get candidates for critical points. The actual critical points are selected from these candidates by evaluating S at each point and eliminating all candidates which do not infer minimum S.

The procedure is made clear by the examples in the following sections.

3.8 Linear Examples

Two examples of the determination of RS(S) for simple systems are given in this section. The region RC is defined by eq. (3-4) and the shape function is the Euclidian norm for both examples.

Example 3-4: The system equations are

$$\frac{d}{dT}\,\underset{\sim}{x}(t) = A\underset{\sim}{x}(t) = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \underset{\sim}{x}(t)$$

Condition (3-9) for this example reduces to

$$2\underset{\sim}{y}^T(T)IA\underset{\sim}{y}(T) = \underset{\sim}{y}^T(T) \begin{bmatrix} 2 & 5 \\ 5 & -8 \end{bmatrix} \underset{\sim}{y}(T) = 0$$

The solution of the above equation by the quadratic formula yields

$$y_1(T) = \frac{-5 \pm \sqrt{41}}{2}\; y_2(T)$$

The transition matrix for the system is

$$\phi(t) = e^{-t} \begin{bmatrix} 3-2e^{-t} & e^{-t}-1 \\ 6-6e^{-t} & 3e^{-t}-2 \end{bmatrix}$$

Substitution of the transition matrix above into equation (3-12) gives

$$-3e^{-T} + 4e^{-2T} = -(-2e^{-2T} + e^{-T})\left(\frac{2}{-5 \pm \sqrt{41}}\right)$$

There are two values of T which satisfy this equation. These are

$$T = \ln \frac{24-4\sqrt{41}}{17-3\sqrt{41}} \quad \text{and} \quad T = \ln \frac{24+4\sqrt{41}}{17+3\sqrt{41}}$$

The corresponding values of b which allow z_1 to be equal to one at these time values are

$$b = -0.151 \qquad\qquad b = 1.325$$

The y(T) candidates are

$$\underset{\sim}{y}(T) = \begin{bmatrix} 0.861 \\ -0.151 \end{bmatrix} \qquad \underset{\sim}{y}(T) = \begin{bmatrix} 0.928 \\ 1.325 \end{bmatrix}$$

The candidate for a critical point which has the smaller value of S is

$$\underset{\sim}{y}(T) = \underset{\sim}{x}_{CRITICAL} = \begin{bmatrix} 0.861 \\ -0.151 \end{bmatrix}$$

The state plane for this system is shown in Fig. 16.

<u>Example 3-5</u>: The following system has complex eigenvalues

$$\frac{d}{dt}\underset{\sim}{x}(t) = A\underset{\sim}{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \underset{\sim}{x}(t)$$

The transition matrix for the system is

$$\phi(t) = e^{-t} \begin{bmatrix} \text{Cos } t + \text{Sin } t & \text{Sin } t \\ -2 \text{ Sin } t & \text{Cos } t - \text{Sin } t \end{bmatrix}$$

Condition (3-9) reduces to

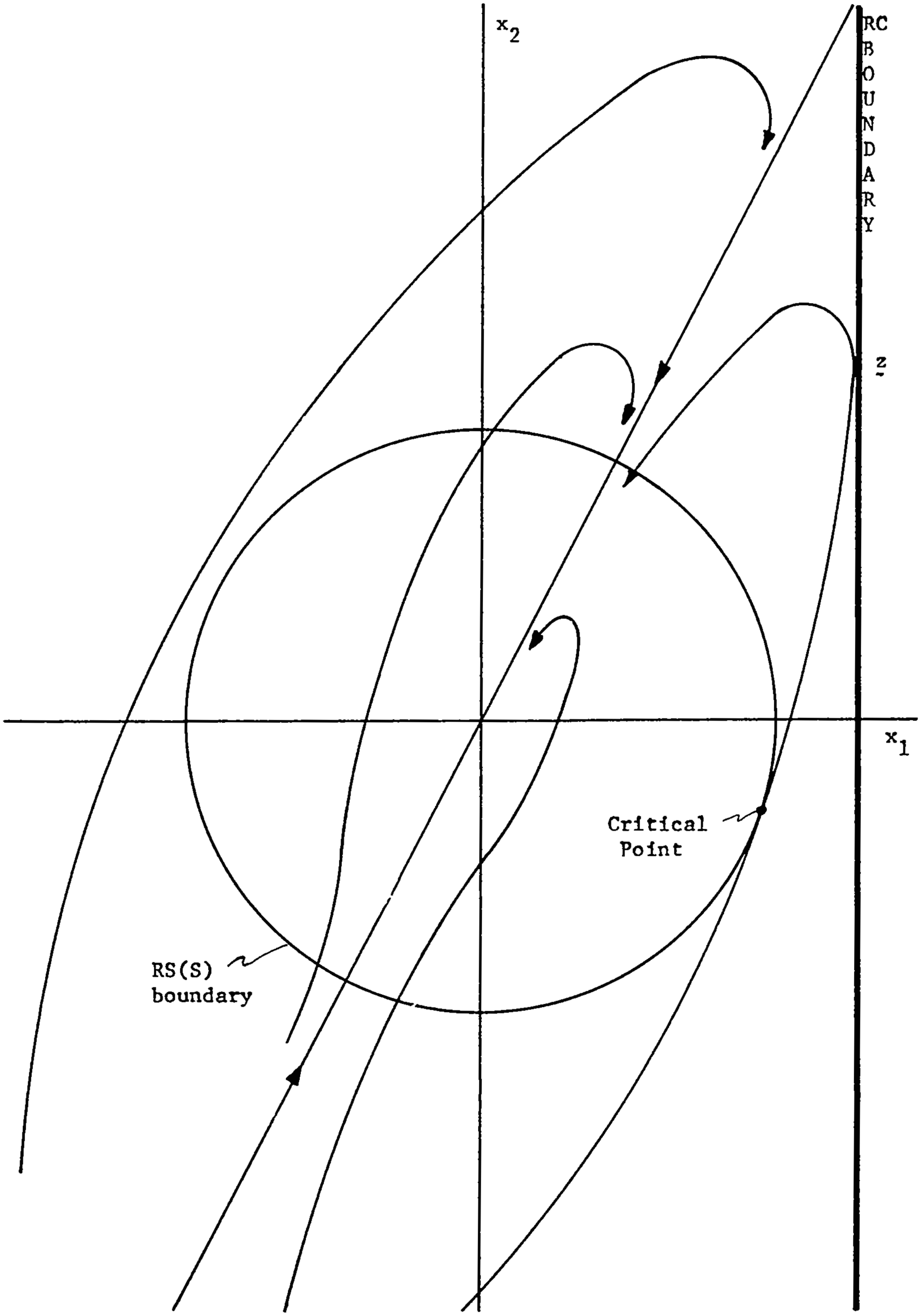

Figure 16. State Plane of Example 3-4

$$2\underset{\sim}{y}^t(t)A\underset{\sim}{y}(T) = \underset{\sim}{y}^T(T) \begin{bmatrix} 0 & -1 \\ -1 & -4 \end{bmatrix} \underset{\sim}{y}(T) = 0$$

which is solved by

$$y_2(T) = 0, y(T) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} b \quad \text{or} \quad y_2(T) = \frac{1}{2} y_1(T),\ \underset{\sim}{y}(T) = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} b$$

Substitution into eq. (3-12) yields

$$3 \sin T + \cos T = 0$$

for the second solution and

$$\sin T = 0$$

for the first solution. These are jointly satisfied by

$$T = \left\{ \begin{array}{l} K\pi \\ \text{P.V. } \tan^{-1}(-\frac{1}{3}) + K\pi \end{array} \right\} \quad K = 0,1,2\ldots$$

The value of the square root of S for successive values of T, taken from the above solution, can be related by the transition matrix

$$S^{1/2}(y(t+\pi)) = ||\,y(t+\pi)\,|| = e^{\pi}\,||y(t)\,||$$

From the above expression it is clear that S increases monotonically with K for the reverse system. Therefore the minimum S-value must occur at a value of T corresponding to K equal zero. There are two values of T when K is zero. The smallest value of S at a critical point candidate evolves when T is zero. The corresponding value of y(T), the critical point, is

$$\underset{\sim}{y}(T) = \underset{\sim}{y}(o) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This example has two rather interesting properties. The one and only critical point and its associated $\underset{\sim}{z}$ point are both the same point, because T is zero. The system has been shown above to be AMS, but the sufficient condition for AMS, eq. (6), is not satisfied. The evaluation of the left hand side of eq. (6) for this example appears below

$$\underset{\sim}{x}^T \begin{bmatrix} 0 & -1 \\ -1 & -4 \end{bmatrix} \underset{\sim}{x} \Big|_{x_1^2 + x_2^2} = -4x_2^2 - 2x_2 \sqrt{1 - x_2^2} \overset{?}{<} 0$$

The condition is violated at $x_2 = -0.1$. The state-plane for this example is shown in Fig. 17.

3.9 An Example with a Limit Cycle

The following example concerns a second-order system with an unstable limit cycle. Limit cycles in two-space divide the space into two complementary sets. The limit cycle in the example bounds a region of asymptotic stability about the origin. Some interesting modifications of the example occur if the RC boundary intersects the limit cycle or if the limit cycle is a stable one.

Example 3-6: Suppose RC and S are given by the same region and function considered in the examples above. The system is described by

$$\frac{d}{dt} \underset{\sim}{x}(t) = \begin{bmatrix} 1 & 1/2 \\ -2 & 1 \end{bmatrix} \underset{\sim}{x}(t) - x(t)g(\underset{\sim}{x}(t))$$

where

$$g(\underset{\sim}{x}) = \begin{cases} (4x_1^2 + x_2^2)^{-1/2} & \text{for } 4x_1^2 + x_2^2 \geq 1 \\ (4x_1^2 + x_2^2) & \text{for } 4x_1^2 + x_2^2 < 1 \end{cases}$$

The origin is asymptotically stable inside an origin-centered ellipse which intersects the x_1 axis at $\pm \frac{1}{2}$ and intersects the x_2 axis at ± 1. Since the RC boundary lies wholly outside of this region of stability, the region RS is the same as the

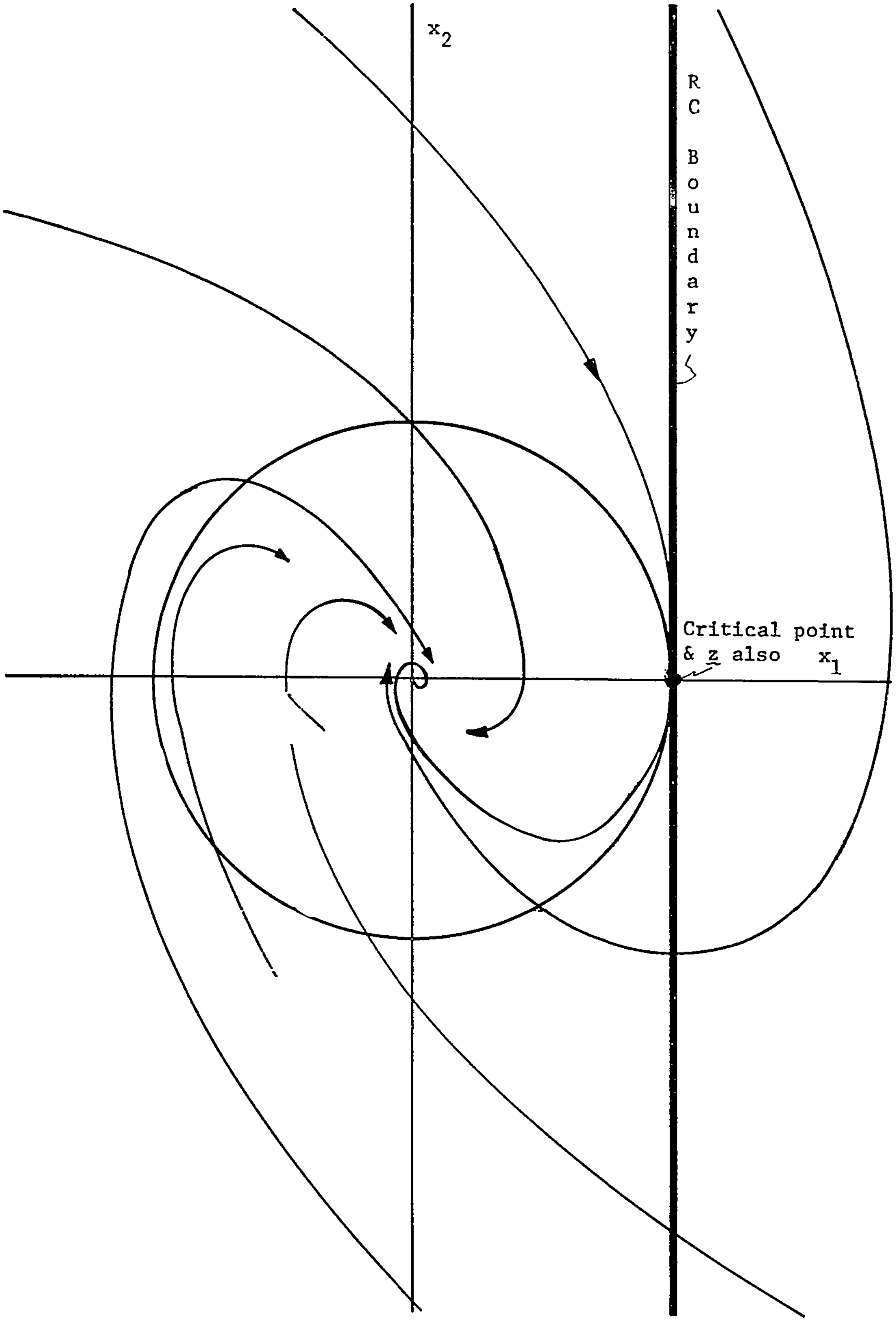

Figure 17. State-Plane of Example 3-5

region of asymptotic stability as indicated on the state plane diagram shown in Fig. 18. The approximate stability region, RS(S), is the largest circle which fits inside this ellipse. RS(S) is bounded by the circle about the origin with radius one half.

The reverse system associated with the example system is described by

$$\frac{d}{dt}\,\underset{\sim}{y}(t) = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & -1 \end{bmatrix} \underset{\sim}{y}(t) + \underset{\sim}{y}(t)g(\underset{\sim}{y}(t))$$

These equations have a closed form solution for trajectories which initiate outside of the limit cycle. The solution is

$$\underset{\sim}{y}(t) = [e^{-t} + \frac{1-e^{-t}}{\sqrt{4z_1^2 + z_2^2}}] \begin{bmatrix} \text{Cos } t & \frac{1}{2}\text{ Sin } t \\ -2\text{ Sin } t & \text{Cos } t \end{bmatrix} \underset{\sim}{z}$$

The example was devised so that a solution would exist. The solution can be verified by differentiation and substitution back into the reverse equations. The existence of the closed form solution makes it possible to analyze the system by a method similar to the method utilized in the previous linear examples.

The solutions of the reverse system which initiate on the boundary of RC are obtained from the general solution above by setting z_1 equal to one. The value of the shape function along reverse trajectories from the RC boundary can be expressed as a function of backward time and the initial value of the unconstrained variable, z_2, by substituting the solution into a formal expression for S as below

$$S(\underset{\sim}{y}) = \underset{\sim}{y}^T\underset{\sim}{y} = \left\{ e^{-t} + \frac{1-e^{-t}}{\sqrt{z_2^2 + 4}} \right\}^2 \left\{ \frac{z_2^2+4}{4} + \frac{3}{4}h\ (z_2,t) \right\}$$

where

$$h(z_2,t) = 4\text{ Sin}^2t - 4z_2\text{Sin } t\text{ Cos } t + z_2^2\text{ Cos}^2t$$

The quantity in the left most brackets of the expression for S is monotone decreasing in time. The quantity in the right hand brackets is periodic with a relative minimum indicated by

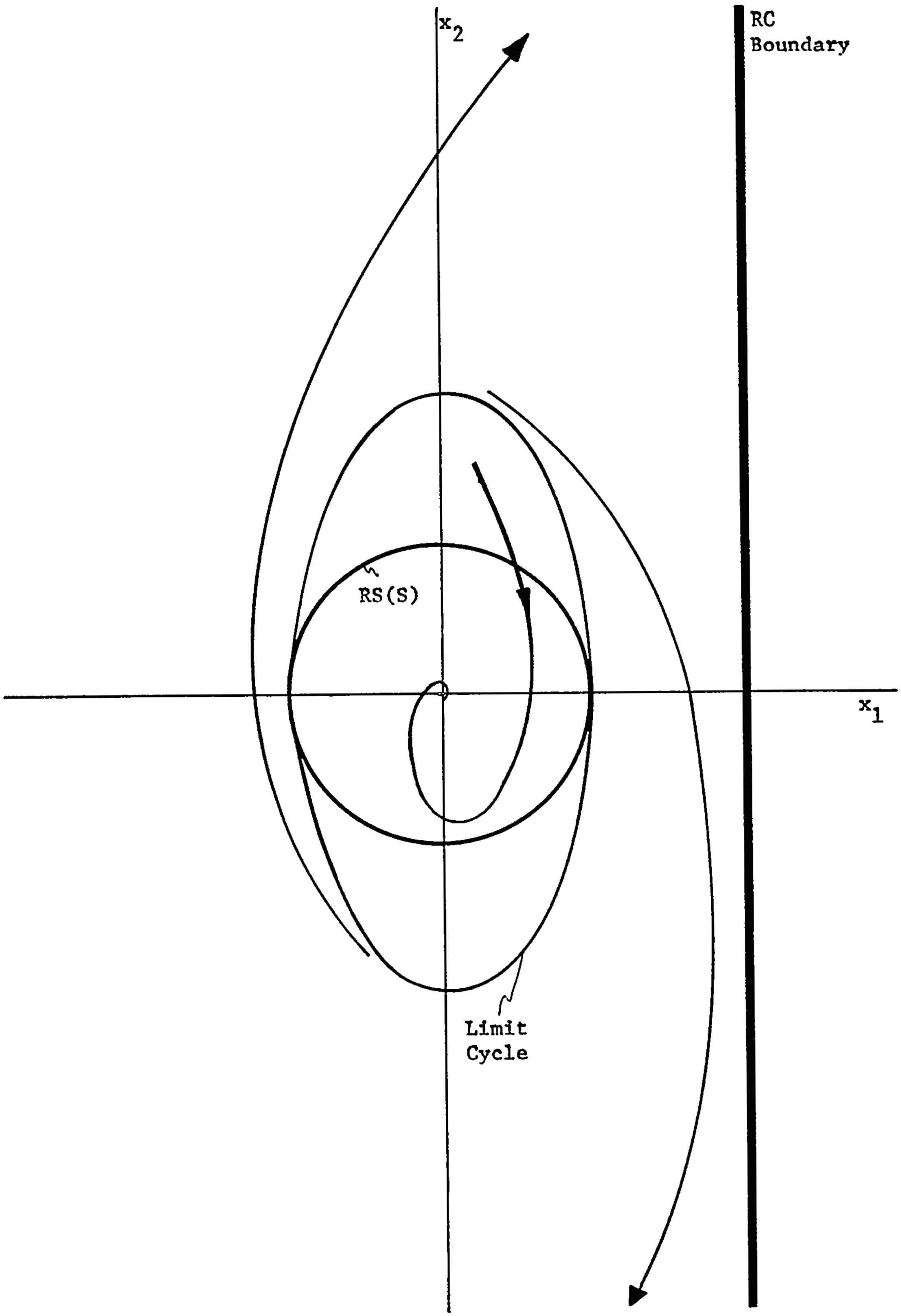

Figure 18. State-Plane of Example 3-6

$$\frac{z_2^2 + 4}{4} \leq \frac{z_2^2 + 4}{4} + \frac{3}{4} h\ (z_2, t)$$

The inequality above may be verified by rewriting h as

$$h(z_2, t) = [2\ \mathrm{Sin}\ t - z_2\ \mathrm{Cos}\ t]^2$$

Now it is clear that

$$h(z_2, t) \geq 0$$

However, whenever t reaches a value of

$$\tau = \tan^{-1} \left(\frac{z_2}{2}\right)$$

then h is exactly zero as shown below

$$h(z_2, \tan^{-1} \frac{z_2}{2}) = [2\ \mathrm{Sin}\ \tau - (2\ \tan\ \tau) \mathrm{Cos}\ \tau]^2 = 0$$

The shape function has a relative minimum at any of these periodic zeros of h, but these minima continue to decrease monotonically approaching

$$S(y) > \left(\frac{1}{\sqrt{z_2^2 + 4}}\right)^2 \frac{z_2^2 + 4}{4} = \frac{1}{4} \qquad (3\text{-}13)$$

as shown in Fig. 19. Clearly the shape function does not have an absolute minimum but it does have the greatest lower bound given above.

A study of the state plane in Fig. 19 shows what is happening. The reverse trajectories are attracted toward the limit cycle. At two points along each encirclement of the limit cycle, near the intersection of the trajectory with the minor axis of the ellipse, the distance between the trajectory and the origin reaches a relative minimum. As the trajectory continually approaches the limit cycle these relative minima approach the minimum distance of the limit cycle itself from the origin, which is realized at x_1 equal one half and x_2 equal zero. Notice also that the same general behavior of reverse trajectories is exhibited regardless of where on the RC boundary

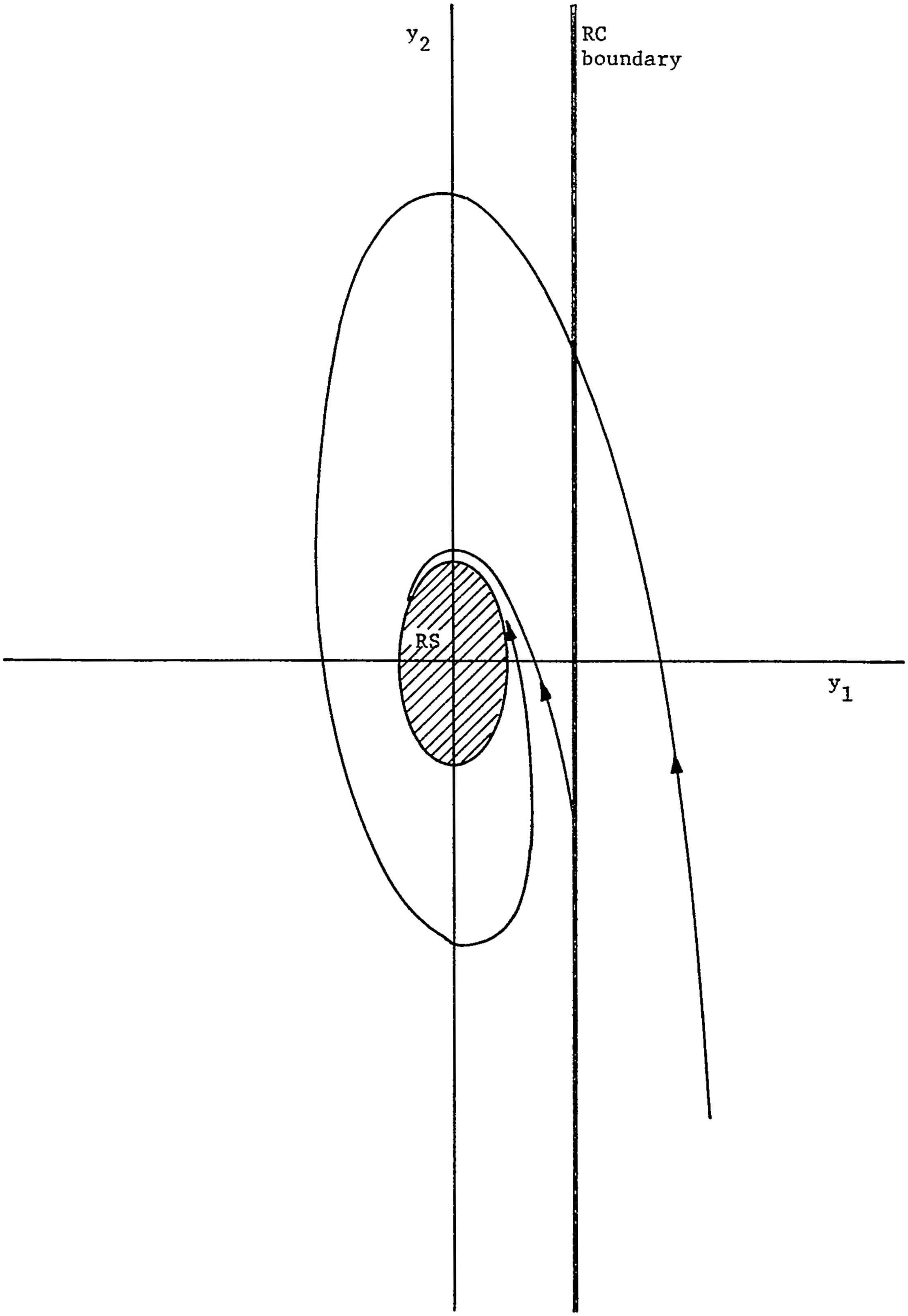

Figure 19. State-Plane for Reverse System in Example 3-6

the trajectory initiates. This observation from the state plane is reinforced by the cancellation of z_2 from eq. (3-13) above.

The following modifications of the example above are noteworthy. If the RC boundary is changed to some other line which lies outside of the limit cycle, then RS and RS(S) remain exactly the same as shown in Fig. 20. All reverse trajectories which start outside the limit cycle approach the limit cycle asymptotically. If the RC boundary intersects the limit cycle, then RS and RS(S) are reduced accordingly, as in Fig. 21. RS does not degenerate into the empty set unless RC does not include the origin because the system is asymptotically stable inside the limit cycle.

If the example system is replaced by a system described by the same equations with the right hand side multiplied by minus one, then the system looks like the reverse system associated with the original example. This new system has a stable elliptic limit cycle which attracts trajectories initiating anywhere in the state plane except the origin. The origin becomes an unstable equilibrium point. An RS is shown in Fig. 22. Trajectories which initiate inside the limit cycle are trapped by the limit cycle. Trajectories initiating outside the limit cycle diverge. In case the limit cycle is not wholly inside RC, then there is no region of excursion stability, as indicated in Fig. 23. All trajectories eventually approach the limit cycle where they must necessarily traverse the RC boundary.

3.10 Asymptotic Excursion Stability

The behavior of a system inside RS is not specified in the definition of the region of excursion stability. A stable limit cycle

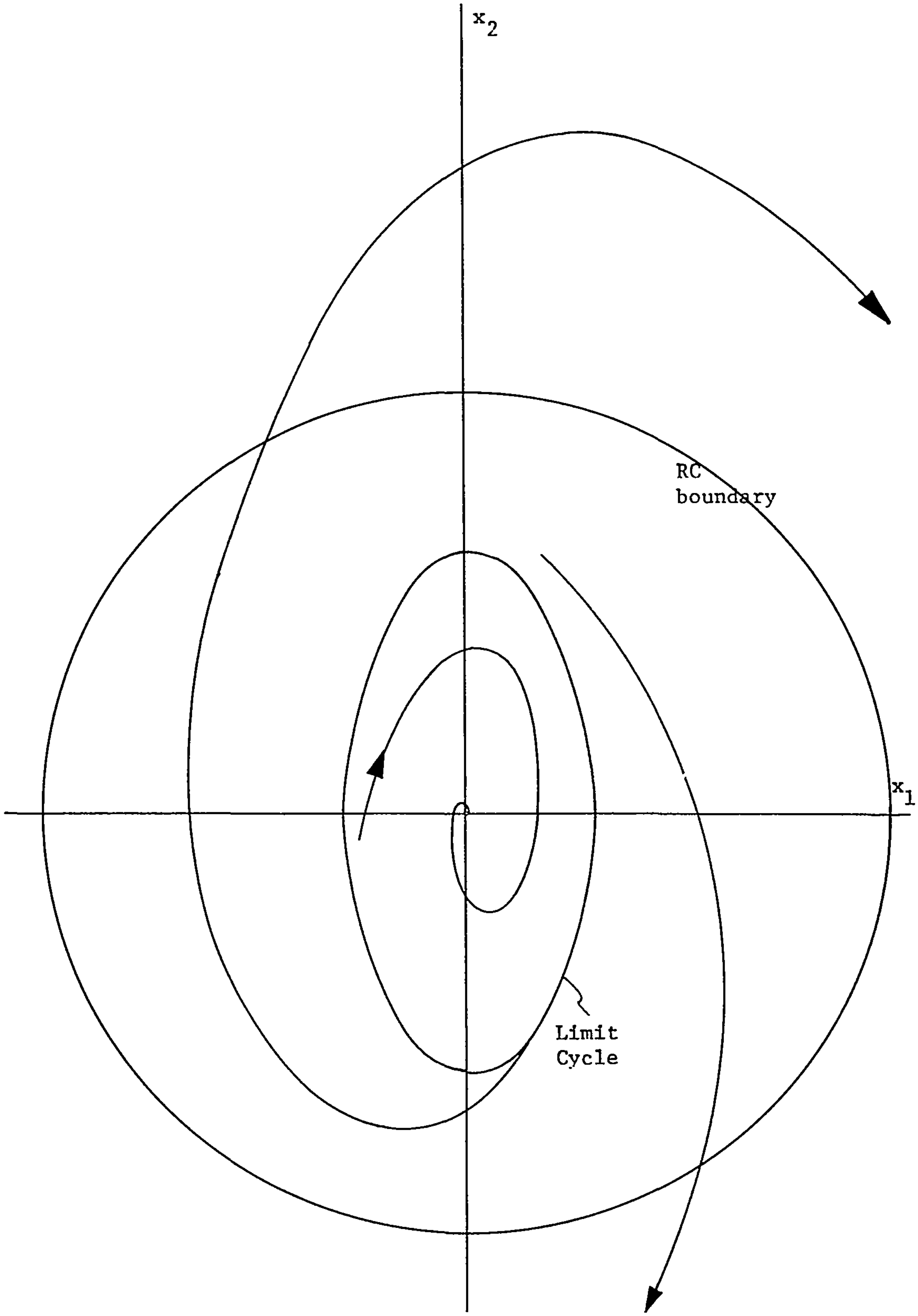

Figure 20. Example 3-6 with Modified RC

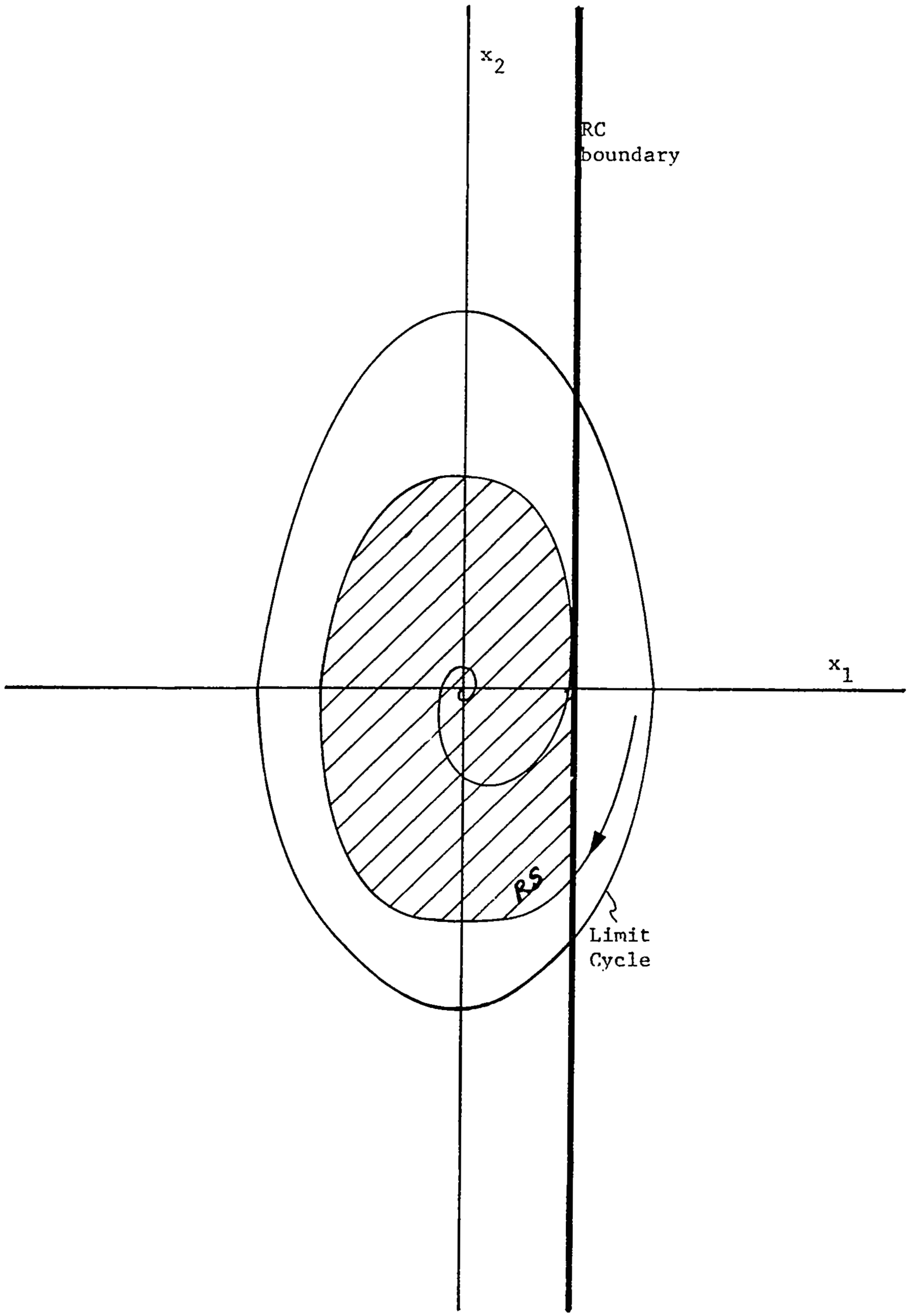

Figure 21. Example 3-6 with a Limiting RC

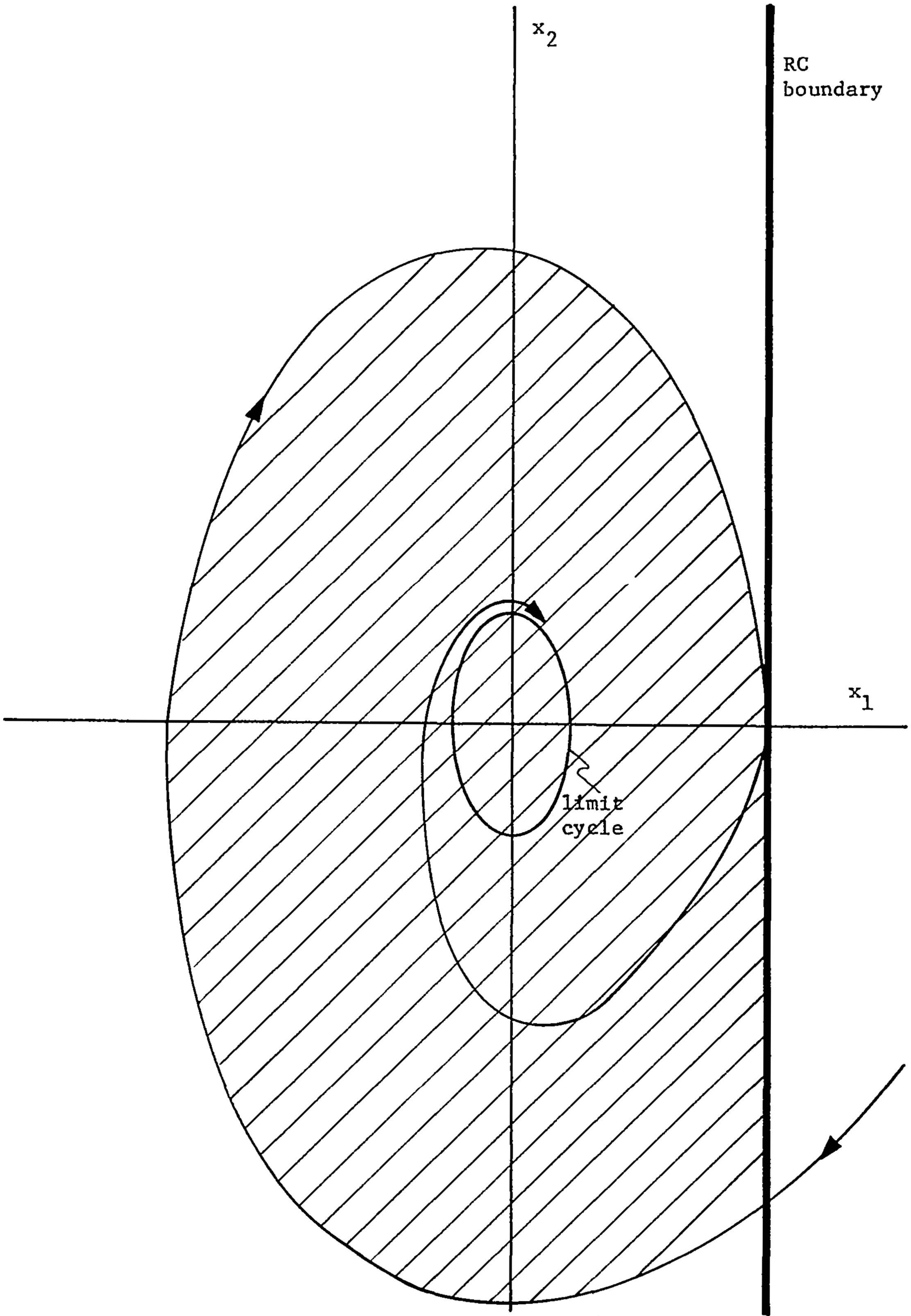

Figure 22. A Stable Modification of Example 3-6

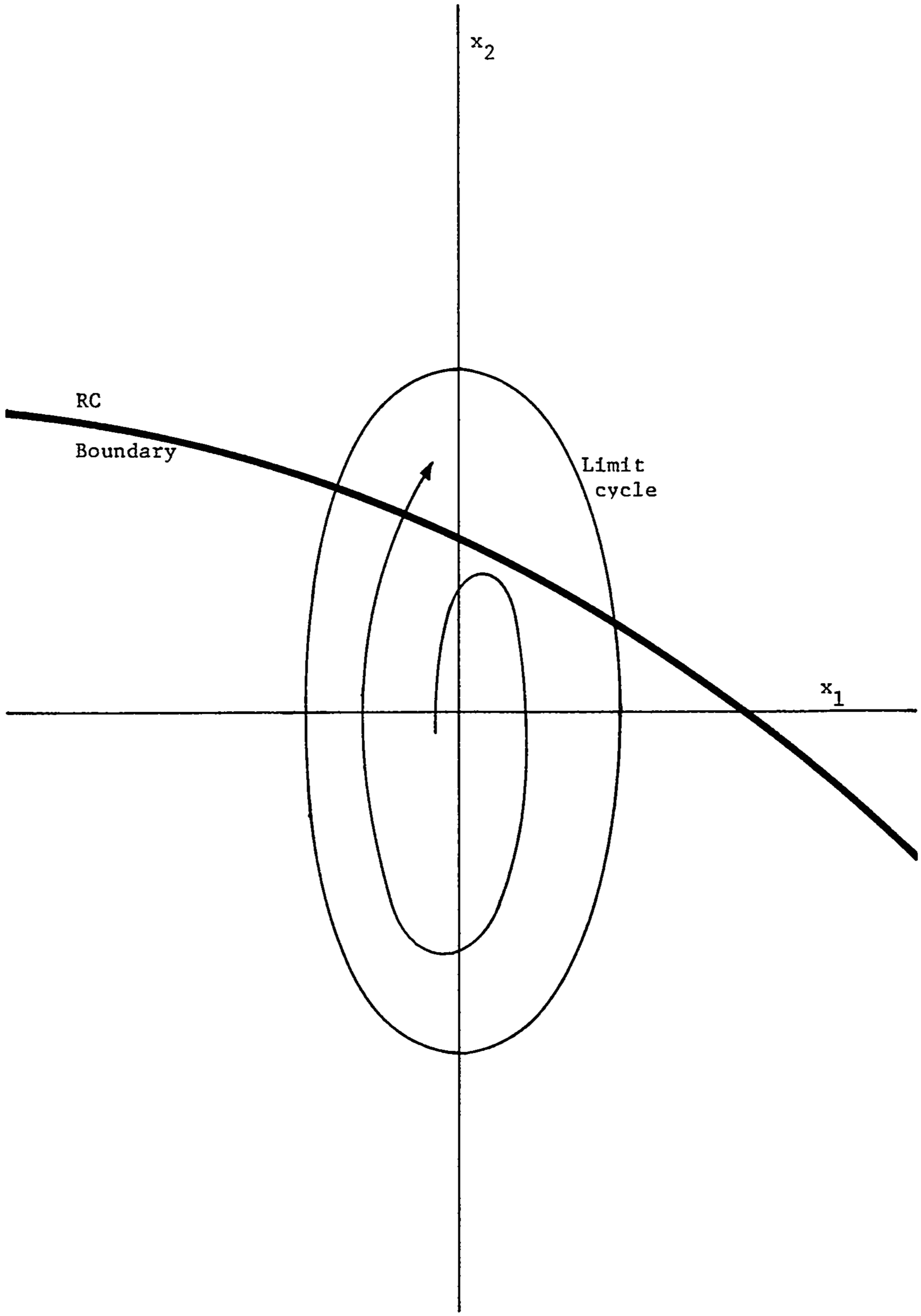

Figure 23. An Unstable Modification of Example 3-6

could exist in RS. There could be a surface contained in RS to which trajectories are attracted. There might be more than one stable equilibrium point in RS. These conditions are objectionable for most systems.

Definition 3-3: Suppose that RC and $\underset{\sim}{q}$ are specified, where RC is a constrained region of operation and $\underset{\sim}{q}$ is a stable equilibrium point. Define the region of asymptotic excursion stability as the set of all points which are excursion stable and in addition require that for any positive number, ε, there is a time increment, Γ, in which the system recovers from any disturbed state in the region of stability in the sense that

$$|| \underset{\sim}{q} - \underset{\sim}{x}(\Gamma) || < \varepsilon$$

Define the approximate region of asymptotic excursion stability as the largest region of prescribed shape which can be inscribed in the region of asymptotic excursion stability.

It is generally not possible to ascertain whether or not the region RS is also asymptotically stable. However, there are a number of steps which can be taken to discover the existence of undesirable behavior in RS. An excellent way to detect nonasymptotic behavior in RS very quickly is simply to test the equilibrium point for asymptotic stability in an arbitrarily small region and then to integrate a few trajectories of the system, numerically or by analog simulation, from arbitrarily selected points in RS. The stability of the equilibrium point can most often be determined by evaluating the eigenvalues of a linearization of the system about the desired equilibrium point, as demonstrated by LaSalle and Lefschetz (1961). If there is a limit cycle or an undesirable equilibrium point condition, one or more of the test trajectories is likely to be attracted by it.

Greater confidence in the lack of nonasymptotic behavior in RS can be enjoyed by recalculating RS(S) with special nonphysical constraints.

Let RC be replaced by a closed bounded region within which hunting can be tolerated. If the region RS(S) still exists, then there is little likelihood of a limit cycle existing in RS. If the region of stability disappears, then the equilibrium point enclosed is either asymptotically unstable or else there is a limit cycle in the original RS which can attract trajectories from points very near the equilibrium point. If desired, more information can be obtained by replacing RC again.

3.11 Conclusion

The special cases of maximal stability and easily demonstrated approximate maximal stability may be encountered in practice from time to time. Generally, however, a given system does not satisfy either of these special categories. Many of the properties of excursion stability have been observed in this chapter by studying simple systems analytically. It is necessary to be able to analyze far more complex systems. The method of analysis used in sections 3.8 and 3.9 was very cumbersome even though the examples all involved only simple systems. Moreover, the method requires a closed loop solution of the system equations. Clearly a more powerful analysis tool is required.

The stability analysis is reformulated into an optimization problem in this chapter. By using this restated problem it is possible to exploit the vast background of optimal control theory, variational calculus and parameter optimization techniques to solve the stability problem. These techniques require the use of a computer so the methods of Chapter Four are presented in the form of computational algorithms.

Chapter Four

COMPUTATIONAL ALGORITHMS

4.1 Introduction

Most nonlinear systems are not susceptible to analysis by the analytical procedure outlined in chapter 3. It is very difficult to apply the method to high-order linear constant systems. The examples of Chapter Three and the method used for their solution are presented to illustrate the nature of the excursion stability analysis problem. Availability of the proper computational facilities makes the analysis of nonlinear systems practicable and the analysis of linear systems easy. Computational algorithms for the analysis of dynamic systems are generated in this chapter.

A complete analysis should contain the following steps. First the system must be modelled and the constraint region and the shape function must be selected by someone who is familiar with the physical system. Equilibrium points in RC can be found by solving the following algebraic equations

$$\underset{\sim}{f}(\underset{\sim}{q},t) \equiv \underset{\sim}{0}$$

The conditions for exhibiting maximal stability and AMS should be applied to test for the existence of either of these two simple cases. Several simulations of the system should be made, preferably on an analog computer, to aid in the location of limit cycles and undisclosed equilibrium points. Finally, RS(S) must be found. The means by which RS(S) is found is the subject of this chapter.

Excursion stability analysis is reformulated in Chapter Three. The reformulated problem is to find a solution of

$$\frac{d}{d\tau}\underset{\sim}{y}(\tau) = -\underset{\sim}{f}(\underset{\sim}{y}(\tau),\tau) \tag{4-1}$$

which minimizes, in a general sense, the shape function

$$S(\underset{\sim}{y}(\tau)) = \underset{\sim}{y}^T(\tau)B\,\underset{\sim}{y}(\tau) \tag{4-2}$$

subject to initial conditions on the RC boundary as given by

$$G(\underset{\sim}{y}(o)) = 0 \tag{4-3}$$

The critical trajectory described above is selected by a seeking method or by a variational technique.

The seeking method recommended is a random search. The random search is discussed in section 4.3. The random search may be applied after the problem is reformulated as a parameter optimization problem as discussed in section 4.2. The alternative variational formulation is presented in terms of the maximum principle in section 4.4. The variational method may be carried out by solving a sequence of two-point boundary-value problems on a set of equations related to the system equations but with twice the order. Two numerical techniques for solving the boundary-value problems are the recirculation algorithm and quasilinearization. The recirculation algorithm is discussed in section 4.5. It is poorly suited to the relevant problem. Quasilinearization which is discussed in section 4.6, worked satisfactorily but was too time consuming to be practical.

4.2 Reformulation as a Parameter Optimization Problem

The selection of the trajectory which satisfies eq.'s (4-1), (4-2) and (4-3) can be recast as a parameter optimization problem. This reformulation is appealing because the parameter selection problem is susceptible to seeking methods or organized search methods. Seeking methods make it possible to avoid the nonlinear boundary-value problems which must be solved to affect the variational methods.

Suppose that the initial conditions of solutions of eq. (4-1) are considered as a vector parameter $\underset{\sim}{z}$ as in

$$\underset{\sim}{y}(o) = \underset{\sim}{z}$$

The optimum parameter is the vector $\underset{\sim}{z}$ which implies a minimum of expression (4-2) subject to

$$G(z) = 0$$

The cost function value associated with a particular selection of z is the minimum value of S along the historical trajectory from that point, as given by

$$\mathrm{COST}(\underset{\sim}{z}) = \min_{\substack{\tau \geq 0 \\ \underset{\sim}{y}(o) = \underset{\sim}{z}}} \{S(\underset{\sim}{y}(\tau))\}$$

If the system is explicitly time dependent, such as

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),t)$$

then the reverse system is given by

$$\frac{d}{dt}\underset{\sim}{y}(\tau) = -\ \underset{\sim}{f}(\underset{\sim}{y}(\tau),\ t_o-\tau)$$

The optimum S is

$$S_{OPT} = \underset{\underset{\sim}{z}}{\text{Min}}\ \underset{t_o}{\text{Min}}\ \underset{\substack{\tau \geq 0 \\ y(o) = \underset{\sim}{z}}}{\text{Min}}\ \{S(\underset{\sim}{y}(\tau))\}$$

The initial time, t_o is simply adjoined to the vector parameter $\underset{\sim}{z}$, and the optimum shape function becomes

$$S_{OPT} = \underset{\underset{\sim}{z}\ U\ t_o}{\text{Min}}\quad \text{COST}(\underset{\sim}{z})$$

The minimization referred to above is not a search for a literal minimum as pointed out in Chapter Three. The search is actually conducted for a greatest lower bound. A reverse trajectory from the RC boundary may intersect the critical point, in which case the minimum of S and the greatest lower bound of S are the same. However, it is possible that no reverse trajectory actually intersects the critical point, such as when a limit cycle exists. If the critical point is not intersected, then it is at least possible to find trajectories which experience S minima as close to the critical value of S as desired.

The search is conducted for a minimum of S on reverse trajectories until one of two stopping conditions is satisfied. The reverse trajectory is observed until the shape function reaches some very large upper limit or until some time limit, T_s is reached. The optimum S becomes

$$S_{OPT} = \underset{\underset{\sim}{z}}{\mathrm{Min}} \quad \underset{\substack{0 \leq \tau \leq T_s \\ \underset{\sim}{G}(\underset{\sim}{z}) = \underset{\sim}{0} \\ S(y(\tau)) \leq S_{Max}}}{\mathrm{Min}} [S(y(\tau))]$$

The minimum literally exists for this closed interval. After the minimum over T_s is found, T_s is replaced by a slightly longer interval. If the minimum S over this longer interval is improved, then T_s is again replaced by a longer interval. Eventually either the difference between successive minima falls within the quantization limits of the machine or else the minimum occurs near the end of the interval and approaches zero. In the latter case, RS(S) is the empty set. In the former case, the S minimum limit as T_s increases is taken as a greatest lower bound for the reverse trajectory under observation. If there are no limit cycles or attracting surfaces in RS, then it is usually not necessary to replace T_s more than once.

4.3 Random Search

The most successful way of minimizing S is to conduct a random search. A random search is an organized trial-and-error process for selecting the optimum value of a set of parameters. Another type of organized search is composed of hill climbing, or gradient, methods, as discussed by Korn (1965).

Random searches are often superior to gradient methods when there are steep cliffs in the cost function. It is necessary to compute the cost function for a trial parameter set only once per trial when random searches are conducted. A random search can easily be modified to search large areas of the parameter space for a global minimum. The procedure outlined below was studied by Mitchel (1964). It was later improved by Bekey (1964) and discussed in some detail by Sabroff et al. (1965).

A random search is conducted by starting with some guessed value for the parameters to be optimized. The cost associated with this guess is computed. A random variation is made in the parameters and the cost function is again computed. If the cost function is increased, then a different random variation is made. If the cost function is reduced, then the initial guess is replaced by the trial parameters and the process is continued.

The random variations are obtained from a source of random numbers which are distributed according to some bell-shaped density function. These random numbers are biased in such a way that variations are more likely to occur in the direction, in parameter space, in which recent success has been enjoyed. The variance of the random numbers is decreased as a local minimum is approached.

The search is conducted in two modes called the local and global modes. The search starts in local mode. After a local minimum has been isolated the search switches to global mode. The global search consists of random trials about the last local minimum in ever increasing variance. The global search continues until an improvement in the cost function is observed. If no improvement is exhibited after a specified number of trials the search is terminated.

The procedure is most clearly discussed by describing the program flow chart in Fig. 24(a) and Fig. 24(b). The symbology used in this

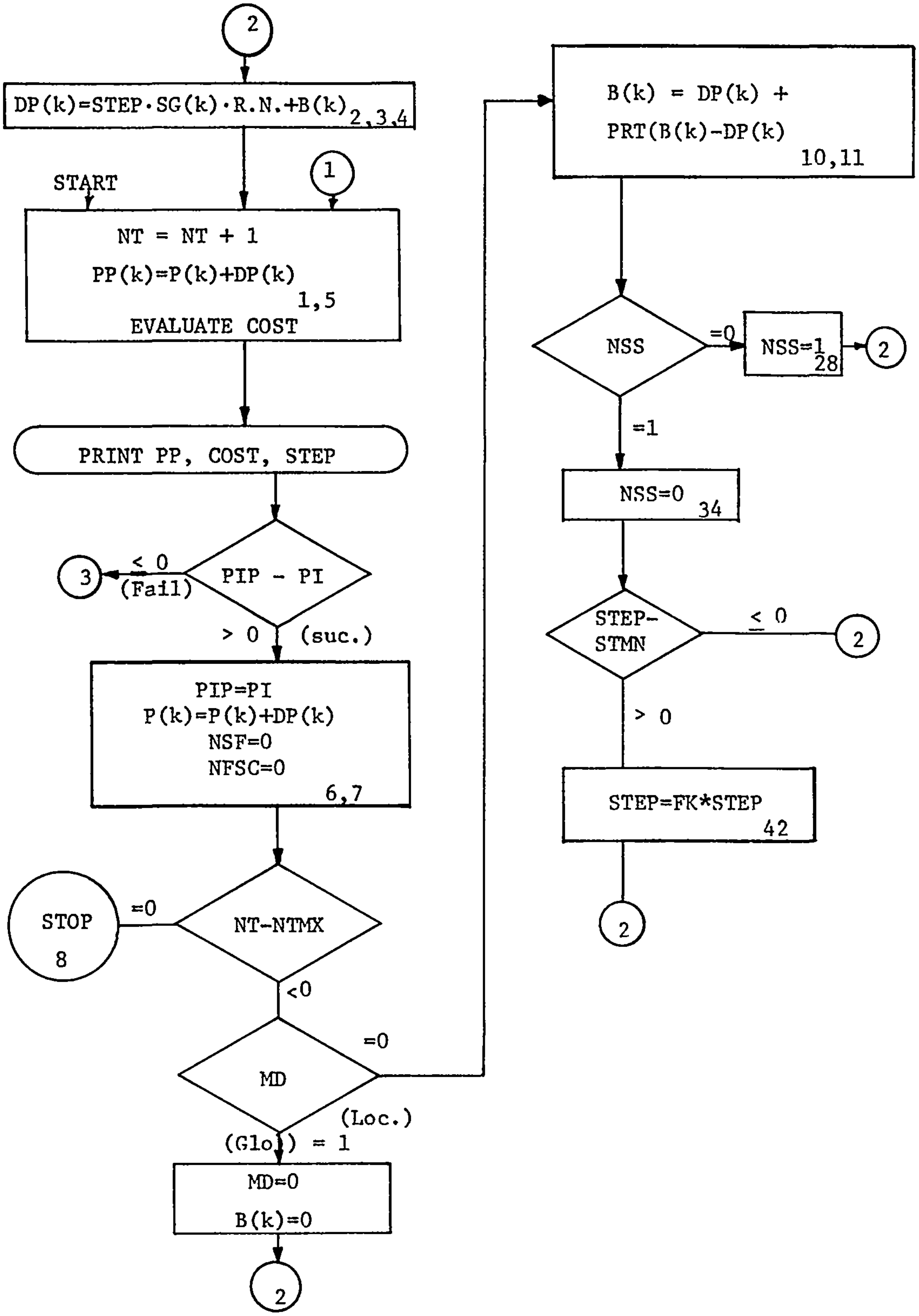

Figure 24(a). Flow Diagram for Random Search Program

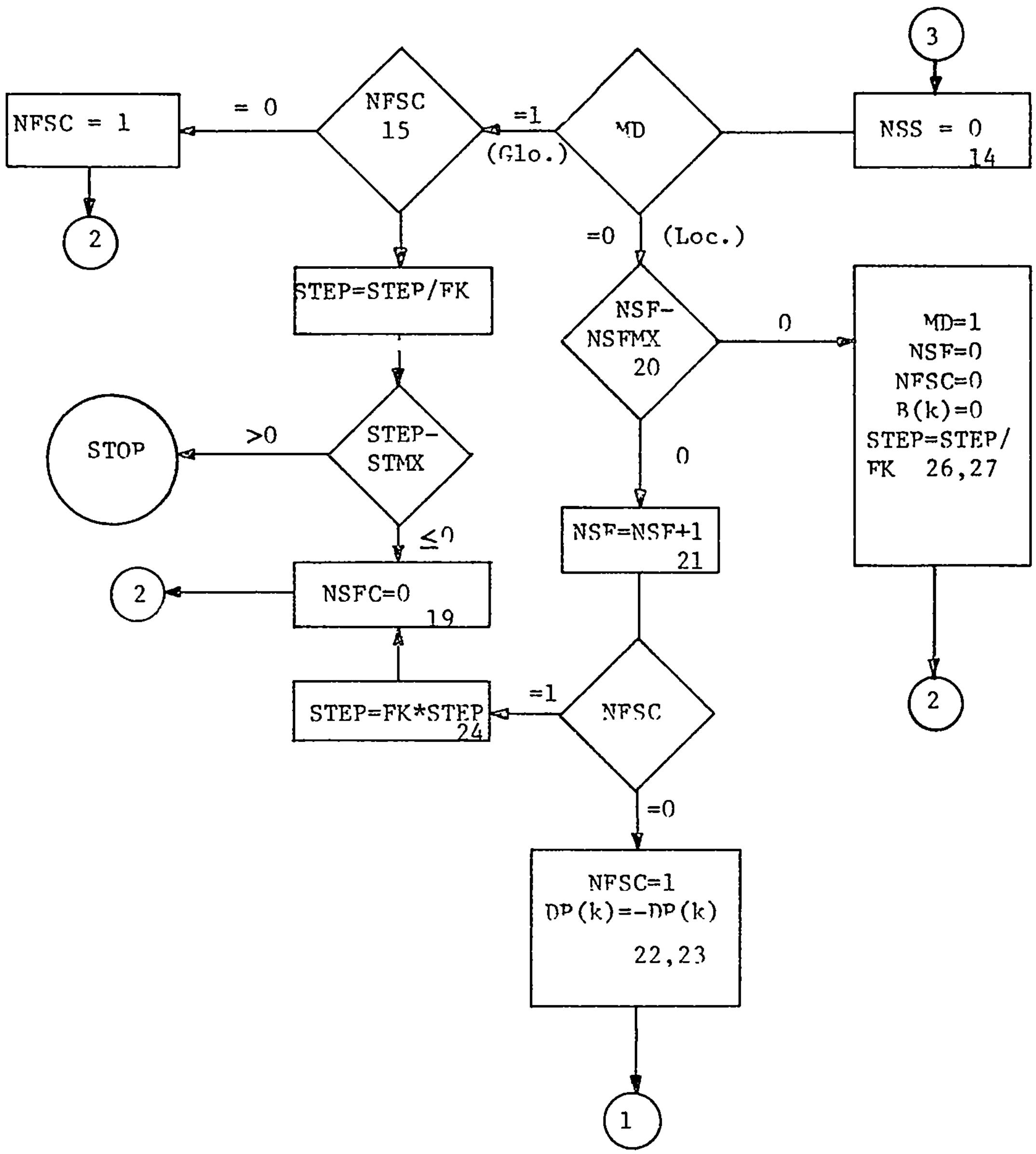

Figure 24(b). Flow Diagram for Random Search Program section does not follow the conventions established in Chapter One. The symbols used below are the same symbols that are used in the Fortran version of the program. NT represents the number of trials. NT is initially set to zero and incremented by one at the beginning of each trial. P(k) is the k^{th} parameter and DP(k) is the trial variation in the k^{th} parameter. The trial parameter is PP(k). The cost function associated with the trial parameters is PI. After PI is evaluated,NT, PI, PP(k), and the Step size, STEP, are printed out on a line printer. The trial cost is compared to the previous best cost, PIP. The program is now directed down two possible subroutines depending upon whether or not the trial parameters caused an improvement in the cost.

In case of an improvement, designated by success, the best cost and parameter combination is brought up to date. The number of successive failures, NSF, is set to zero. The number of successive failures since a change of variance, NFSC, is also set equal to zero. If there have not already been too many trials, the mode index MD is tested. MD is set equal to zero in the local mode and it is set equal to one in the global mode. If the recent success was achieved in the global mode, then the mode is changed to local and the bias, B(k), is removed. A unit variance and zero mean random number, RN, is read in for each parameter. A new trial parameter set is selected and the cost evaluated. If, alternatively, the success was achieved in the local mode, then the noise bias is brought up to date. The number of successive successes since a change in step size, NSS, is tested. If NSS is zero, it is set equal to one and a new trial is generated. If NSS is one, it is set equal to zero and the step size is reduced by the factor, FK, unless it has already been reduced past the minimum step size.

In case the trial cost function fails to improve, then NSS is zeroed and the mode index is tested. If the failure occurred in the local mode, then the subsequent steps depend on whether or not the maximum number of successive failures (NSFMX) necessary to switch to global search have been achieved. If there have been enough failures, the mode is changed to global and the step size is increased. If there have not been too many failures, then NSF is incremented and the number of failures since changing the step size is tested. If NFSC is zero, then it is set equal to one and the negative of the old trial variation is tried. If NFSC is one, then the step size is reduced and a new trial step generated. Alternatively, if the failure occurred in the global mode, then NFSC is tested. If NFSC is zero a new trial is generated. If NFSC is one, the step size is increased, up to a maximum, before generating a new trial. If the step size is maximum, then the program is terminated.

The cost function is evaluated by integrating the reverse equations from the initial state chosen by the random number generator. These numbers are generated and selected so that the initial states lie on the RC boundary. The shape function is evaluated at each step of the reverse equation integration and the minimum value is retained until the shape function gets very large or else until the equations have been integrated for time T_s. The minimum value of S over this interval is assigned to the cost function.

Consider an n^{th}-order problem with an inequality constraint on x_1. The variable $y_1(0)$ is set equal to the extreme value of x_1. All of the remaining initial conditions on $\underset{\sim}{y}$ are considered to be parameters. There are n-1 parameters. If the system is explicitly time dependent, then the initial time constitutes another parameter. The entire random search procedure outlined above may be carried out solely on a digital computer. The computer spends most of its time integrating the reverse equations as necessary to evaluate the cost function. If the system order is greater than three, then the run time becomes prohibitively large for a digital simulation. This limitation can be circumvented if a hybrid digital-analog computer is available. A hybrid computer consists of both an analog and digital computer operating concurrently and each capable of communicating with the other. The reverse equations may be integrated quite rapidly on the analog portion and the shape function may be computed on the analog computer and monitored on the digital portion. The remainder of the program can be carried out on the digital portion. The analog computer usually has a source of random noise. The random noise source makes it unnecessary to precompute the random parameter perturbations and store them as is necessary in a totally digital operation.

If the cost function is much more sensitive to one group of parameters than another, then the search should be conducted alternately on each group of parameters with the other group held constant at the most recently obtained value. The problem of widely varying sensitivity is not unique to the parameter search. However, the problem is less troublesome with seeking methods than it is with other methods.

4.4 Maximum Principle

The transformed problem of excursion analysis as discussed in section 4.1 may alternatively be reformulated as a variational problem. The variational methods suggest some interesting modifications of the analysis. Unfortunately, the variational formulation leads to a sequence of boundary-value problems which cannot be solved efficiently on the computer.

The variational formulation is discussed in terms of the maximum principle because this is the formulation which leads to the modified analysis. Dynamic programming is not considered because it is basically an exhaustive search and the memory requirements increase very rapidly with the order of the system under study.

The variational problem is to select a solution of eq. (4-1) subject to eq. (4-3) which minimizes expression (4-2) at the terminal time T. The Pontryagin state function is given by

$$H(\underset{\sim}{y}(\tau),\ \underset{\sim}{p}(\tau),\tau) = -\ \underset{\sim}{p}^T(\tau)\underset{\sim}{f}(\underset{\sim}{y}(\tau),\tau)$$

The reverse equations may be expressed in terms of this state function as

$$\frac{d}{d\tau}\underset{\sim}{y}(\tau) = \frac{\partial}{\partial p} H(\underset{\sim}{y}(\tau),\ \underset{\sim}{p}(\tau),\tau)$$

The adjoint variables $\underset{\sim}{p}$ are defined by solutions of the following ordinary differential equation

$$\frac{d}{d\tau}\underset{\sim}{p}(\tau) = -\frac{\partial}{\partial y} H(\underset{\sim}{y}(\tau),\ \underset{\sim}{p}(\tau),\tau)$$

According to the maximum principle as discussed in Schultz and Melsa, (1967), the optimum trajectory is a solution of a two-point boundary-value problem involving the reverse system equations (4-1), and the adjoint equations which are presented below in accordance with the definition above.

$$\frac{d}{d\tau}\, \underset{\sim}{p}(\tau) = J_f^{\,T}(\underset{\sim}{y},\tau)\underset{\sim}{p}(\tau)$$

The boundary values are determined from the given boundary specifications, eq. (4-3), and the satisfaction of the transversality conditions

$$[\nabla_y G(\underset{\sim}{y}(o)-p(o)]^T Sy\Big|_{\tau=0} + H(\underset{\sim}{y}(o),\ \underset{\sim}{p}(o),\ o) = 0 \quad \text{and}$$

$$[\nabla_y S(\underset{\sim}{y}(T)) - \underset{\sim}{p}(T)]^T Sy\Big|_{\tau=T} + H(\underset{\sim}{y}(T),\ \underset{\sim}{p}(T),\ T) = 0$$

The time interval is fixed in the problem of interest here so that the transversality conditions reduce to

$$\nabla_y G(\underset{\sim}{y}(\tau))\Big|_{\tau=0} = \underset{\sim}{p}(o)$$

at the initial time and

$$\nabla_y S(\underset{\sim}{y}(\tau))\Big|_{\tau=T} = \underset{\sim}{p}(T)$$

at the terminal time.

The solution of the two-point boundary-value problem described above yields the trajectory which minimizes the shape function at the terminal time. This selection is in contrast to the cost function used with the seeking method. The cost function values the minimum of S over an entire interval from zero to T_s. The variational performance index is not necessarily a monotonically decreasing function of T. The best performance index for any T within the interval from zero to T_s is the cost at T_s. The relationships between the variational performance index denoted S_{var} and the optimum cost function denoted by S_{cost} is indicated by Fig. 25. S_{cost} is the quantity which is ultimately of interest. S_{cost} can be obtained from variational solutions by dividing the interval T_s into M increments indexed by k and expressed as

$$T = K \frac{T_s}{M} \quad K = 1, 2, \ldots, M$$

The sequence of variational problems discussed above presents a sequence of two point boundary value problems. The solutions of the boundary value problem are not necessarily unique and they represent necessary but not sufficient conditions for a minimum. A global optimum can be selected with confidence only if all the solutions of the boundary value problem are isolated. Since the boundary value problems involve the reverse equations and the adjoint equations the order of the boundary value problem is two times n. These accumulated difficulties make the computational task associated with the variational methods prohibitively severe.

The maximum principle formulation is usually associated with the selection of an optimum open loop forcing function. The analog of

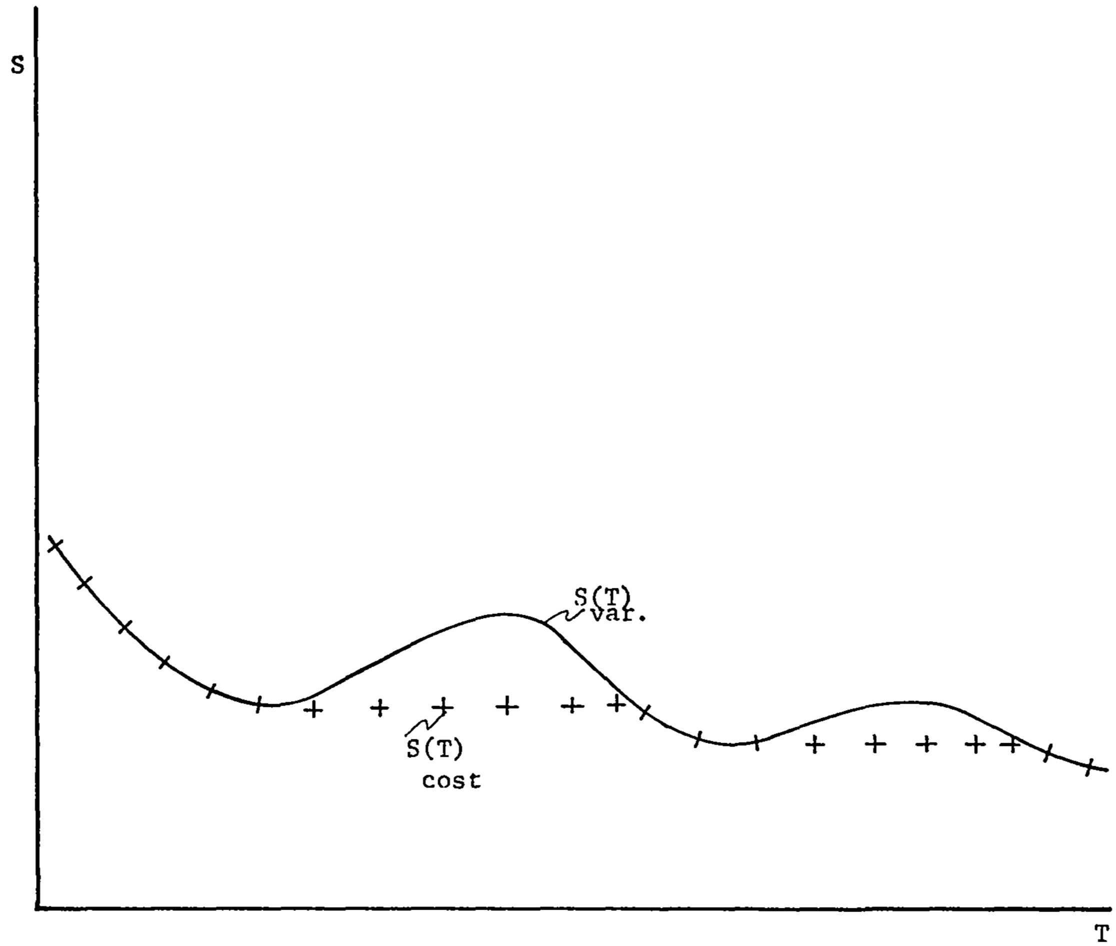

Figure 25. Typical Variational Performance Index and Cost Function versus Duration

the forcing function does not appear in the analogous variational problem discussed above. If the analog of the forcing function is inserted into the formulation above, then it plays the part of a disturbing influence which reduces the extent of the stability region. This modification of the variational problem suggests possibility of analyzing the stability of a system when it is under the influence of some constrained forcing function which is not described even in a stochastic sense. This possibility is considered further in Chapter Six.

4.5 Recirculation Algorithm

There are two well-known techniques for obtaining numerical solutions to nonlinear boundary-value problems. Any hope of solving the stability problem in terms of the variational formulation rests on the success of one of these algorithms. The algorithm discussed in the following section is usable but impractically inefficient. The recirculation algorithm discussed in this section does not converge at all when the time increment gets at all large, even if a very good initial guess is "planted" into the initialization of the algorithm.

The recirculation algorithm is conducted by integrating the differential equations involved back and forth in time between the initial and terminal times. At the end of each cycle of this process enough state variable values are discarded to make it possible to satisfy the relevant terminal condition. The discarded values are replaced by values which satisfy the terminal condition and the process is continued.

The error analysis below indicates that the algorithm can be expected to diverge on long interval problems even if the initial guess is a very good one and the integration subroutine is assumed errorless. The analysis is conducted on a linear system, but the implications are clear, and they agree in nature with the lack of success experienced when the algorithm was applied to nonlinear problems.

Consider the boundary value problem involving

$$\frac{d}{dt}\,\underset{\sim}{x}(t) = A(t)\underset{\sim}{x}(t)$$

where the boundary values are specified as

$$\underset{\sim}{x}_a(o) = \underset{\sim}{a} \quad \underset{\sim}{x}_b(T) = \underset{\sim}{b} \quad \underset{\sim}{x}(t) = \left[\begin{array}{c} x_a(t) \\ \hline x_b(t) \end{array}\right]$$

and $\underset{\sim}{x}_a$ and $\underset{\sim}{x}_b$ are partitions of $\underset{\sim}{x}$ whose dimensions are p by one and q by one respectively. Define the transition matrix as the solution of

$$\frac{d}{dt}\,\phi(t) = A(t)\phi(t)$$

subject to

$$\phi(o) = \text{identity}$$

If the transition matrix is evaluated at the terminal time and partitioned according to the partitions in $\underset{\sim}{x}$, then the initial and terminal values of $\underset{\sim}{x}$ are related by

$$\underset{\sim}{x}(T) = \phi(T)\underset{\sim}{x}(o) = \left[\begin{array}{c|c} \phi_1 & \phi_3 \\ \hline \phi_4 & \phi_2 \end{array}\right] \left[\begin{array}{c} x_a(o) \\ \hline x_b(o) \end{array}\right]$$

where ϕ_1 is p by p, ϕ_2 is q by q, ϕ_3 is p by q, and ϕ_4 is q by p. The aspired value of the unspecified initial conditions are

$$\underset{\sim}{x}_b(o) = \underset{\sim}{c} = \phi_2^{-1} [\underset{\sim}{b} - \phi_4\underset{\sim}{a}]$$

Let H denote the inverse of $\phi(T)$. Partition H similar to ϕ as below

$$\phi^{-1}(T) = H(T) = \left[\begin{array}{c|c} H_1 & H_3 \\ \hline H_4 & H_2 \end{array}\right]$$

After one half cycle of the recirculation algorithm with the unspecified initial conditions estimated as $\underset{\sim}{c}_k$, the resulting system state is

$$\left[\begin{array}{c} \phi_1\underset{\sim}{a}+\phi_3\underset{\sim}{c}_k \\ \hline \phi_4\underset{\sim}{a}+\phi_2\underset{\sim}{c}_k \end{array}\right] = \left[\begin{array}{c} \underset{\sim}{x}_a(T) \\ \hline \underset{\sim}{x}_b(T) \end{array}\right]$$

Now the lower part of $\underset{\sim}{x}$ is discarded, according to the algorithm, and replaced by the correct terminal condition so that

$$\left[\begin{array}{c} \underset{\sim}{x}_a(T) \\ \hline \underset{\sim}{x}_b(T) \end{array}\right] = \left[\begin{array}{c} \phi_1\underset{\sim}{a}+\phi_3\underset{\sim}{c}_k \\ \hline \underset{\sim}{b} \end{array}\right]$$

After performing the backward integration of the last half of one cycle of the algorithm, the resulting system state is

$$\left[\begin{array}{c} \underset{\sim}{x}_a(o) \\ \hline \underset{\sim}{x}_b(o) \end{array}\right] = \left[\begin{array}{c} H_1\phi_1\underset{\sim}{a}+H_1\phi_3\underset{\sim}{c}_k+H_3\underset{\sim}{b} \\ \hline H_4\phi_1\underset{\sim}{a}+H_4\phi_3\underset{\sim}{c}_k+H_2\underset{\sim}{b} \end{array}\right]$$

At this point of the procedure the upper part of the state vector is replaced by $\underset{\sim}{a}$ so that the next guess at $\underset{\sim}{c}$ is

$$\underset{\sim}{c}_{k+1} = H_4\phi_1\underset{\sim}{a} + H_4\phi_3\underset{\sim}{c}_k + H_2\underset{\sim}{b}$$

If the error is defined as

$$\underset{\sim}{\delta}_k = \underset{\sim}{c}_k - \underset{\sim}{c} = \underset{\sim}{c}_k - \phi_2^{-1}[\underset{\sim}{b} - \phi_4\underset{\sim}{a}]$$

then the k+1 st error is

$$\underset{\sim}{\delta}_{k+1} = \underset{\sim}{c}_{k+1} - \underset{\sim}{c}$$

This error can be expressed in terms of the transition matrix by using the relationship between $\underset{\sim}{c}_{k+1}$ and $\underset{\sim}{c}_k$. The expression is

$$\underset{\sim}{\delta}_{k+1} = [H_4\phi_1 - H_4\phi_3\phi_2^{-1}\phi_4 + \phi_3^{-1}\phi_4]\underset{\sim}{a} + [H_2 - \phi_2^{-1} + H_4\phi_3\phi_2^{-1}]\underset{\sim}{b} + H_4\phi_3\underset{\sim}{\delta}_k$$

By the definition of H

$$H\phi = \left[\begin{array}{c|c} H_1 & H_3 \\ \hline H_4 & H_2 \end{array}\right] \left[\begin{array}{c|c} \phi_1 & \phi_3 \\ \hline \phi_4 & \phi_2 \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

The lower left and lower right hand parts of the product can be transposed to show that the coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ in the error expression are both identically zero. The error is therefore

$$\underset{\sim}{\delta}_{k+1} = H_4\phi_3\underset{\sim}{\delta}_k = B(T)\underset{\sim}{\delta}_k$$

Convergence of the error sequence to the origin indicates convergence of the algorithm for the given boundary value problem.

There are some interesting consequences of the error recursion relation above. Since the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ cancelled out of the expression, the specified boundary values do not influence convergence. The recursion formula above is a linear finite difference equation, so that the convergence of the error sequence does not depend on where the sequence initiates. In other words, it does not matter what the boundary values are or how bad the initial guess is, convergence of the recirculation algorithm depends only on the interval specified.

The recirculation algorithm diverges for some simple problems. Example 4-1 is a boundary value problem involving a simple linear time-invariant second-order system with real eigenvalues. The boundary conditions are of the simplest possible type and the solution exists for all time increments. However, the recirculation algorithm can be expected to converge only when the time interval is less than about seven-tenths of a second.

<u>Example 4-1</u>: The system of equations

$$\frac{d}{dt}\,\underset{\sim}{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \underset{\sim}{x}(t)$$

has the transition matrix

$$\phi(t) = \begin{bmatrix} 2-e^{-t} & 1-e^{-t} \\ 2e^{-t}-2 & 2e^{-t}-1 \end{bmatrix} e^{-t}$$

The one-by-one matrix B(T) is

$$B(T) = 2e^{T} - 4 + 2e^{-T}$$

The error recursion formula is

$$\underset{\sim}{\delta}_{k+1} = [2e^{T} = 4 + 2e^{-T}]\underset{\sim}{\delta}_{k}$$

Convergence is indicated if and only if

$$|\ B(T)\ | < 1 \iff T < \ln 2$$

The time interval for susceptible problems is

$$0 \leq T < \ln 2 \approx 0.694$$

B(T) is the ratio of successive errors in the estimate of $x_2(0)$ when the time increment is T. B is plotted in as a function of T in Fig. 26. Figure 26 also contains a plot of the minimum number of iterations necessary to reduce the error by a factor of ten as a function of time.

4.6 Quasilinearization

Quasilinearization is another method for the numerical solutions of boundary value problems. It is attributed to Kalaba (1959), and it is the subject of a book by Bellman and Kalaba (1965). It has been used by Cox (1964) in the derivation of a sequential least-square filter and also by O'Hap and Stubberund (1965) in the development of a nonlinear stochastic filter. It is basically a linearization about a whole solution of the nonlinear equations involved in the boundary value problem. Therefore, it is limited more by the quality of the initial estimate than it is by the duration of the problem. Hence, it is more suitable, for the problem of interest in this work, than the recirculation algorithm.

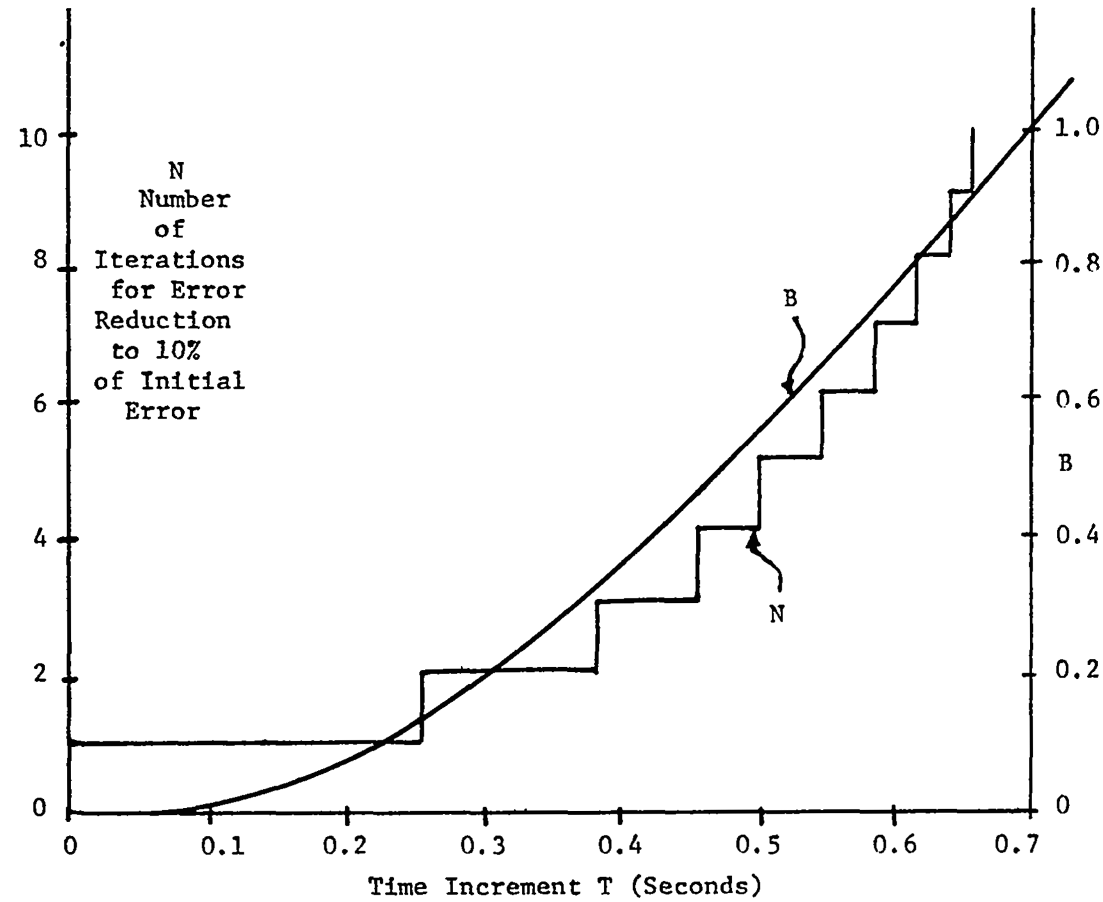

Figure 26. Successive Error Ratio and Number of Iterations for 90% Error Reduction

The method requires the repeated numerical integration of a set of differential equations on known initial conditions. Unfortunately, these related equations are of much higher order than the system under study. Consequently, the algorithm taxes the speed capability of a digital computer. The solution of the third order example in Chapter Five illustrates the inferiority of quasilinearization to the random search with regard to computational efficiency. However, linear systems may be analyzed quite readily by the quasilinearization program.

Let

$$\frac{d}{dt}\,\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x},\ t) \qquad (4\text{-}4)$$

represent a system of equations involving an n-vector $\underset{\sim}{x}$, upon which noncon-current boundary values are imposed. Suppose $\underset{\sim}{x}^k(t)$ is the k^{th} estimate of the desired solution. Let the estimates corresponding to odd k be solutions to the nonlinear equations (4-4). Let the even numbered estimates be solutions of the quasilinear equations defined by

$$\frac{d}{dt}\,x^{k+1}(t) = J_f(\underset{\sim}{x}^k,t)[\underset{\sim}{x}^{k+1}(t)-x^k(t)] + \underset{\sim}{f}(\underset{\sim}{x}^k,t) \qquad (4\text{-}5)$$

where J is the Jacobian of f. Equation (4-5) is a linear time-varying equation in $\underset{\sim}{x}^{k+1}$ with a solution given by

$$\underset{\sim}{x}^{k+1}(t) = \phi^k(t)\underset{\sim}{x}^{k+1}(o) + \underset{\sim}{w}^k(t)$$

where $\phi(t)$ is the transition matrix and solves

$$\frac{d}{dt}\,\phi^k(t) = J_f(\underset{\sim}{x}^k,t)\phi^k(t)$$

subject to

$$\phi^k(o) = \text{Identity Matrix}$$

and $\underset{\sim}{w}^k(t)$ is the particular solution of eq. (4-5) subject to

$$\underset{\sim}{w}^k(o) = \underset{\sim}{0}$$

Suppose boundary conditions are specified at t = 0 and t = T. The final and initial states of the quasilinear variables are related by the solution of the quasilinear equations as given by

$$\underset{\sim}{x}^{k+1}(T) = \phi^k(T)\underset{\sim}{x}^{k+1}(o) + \underset{\sim}{w}(T) \tag{4-6}$$

Equation (4-6) may be solved for the unknown boundary conditions once ϕ and $\underset{\sim}{w}$ are determined.

The procedure is to alternately generate solutions of the non-linear equations and solutions of the quasilinear equations which satisfy the boundary conditions. The former approach the desired solution to the boundary value problem.

If the system is linear eq. (4-4) reduces to

$$\frac{d}{dt}\underset{\sim}{x}^k(t) = A(t)\underset{\sim}{x}^k(t)$$

and eq. (4-5) becomes

$$\frac{d}{dt}\underset{\sim}{x}^{k+1}(T) = A(T)[\underset{\sim}{x}^{k+1}(t)-\underset{\sim}{x}^k(t)] + A(t)\underset{\sim}{x}^k(t) = A(t)\underset{\sim}{x}^{k+1}(t)$$

Since the linear and quasilinear equations are identical, both have the same solution. Therefore, linear systems are susceptible to the method in exactly one iteration. Convergence is not guaranteed for nonlinear problems. However, the algorithm typically converges or diverges in a few iterations.

Consider a system described by

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),t)$$

where RC is given by

$$RC = \{ \underset{\sim}{x} \mid x_1 \leq M \}$$

and the shape function is given by

$$S(\underset{\sim}{x}) = \underset{\sim}{x}^T B \underset{\sim}{x}$$

where B is a constant symmetric matrix. The relevant Euler-Lagrange equations as presented in section 4.4 can be represented by a partitioned equation

$$\frac{d}{dt}\left[\begin{array}{c} \underset{\sim}{y}^k(t) \\ \hline \underset{\sim}{p}^k(t) \end{array}\right] = \left[\begin{array}{c|c} -I & 0 \\ \hline 0 & J_f^T(\underset{\sim}{y},t) \end{array}\right] \left[\begin{array}{c} f(\underset{\sim}{y}^k(t),t) \\ \hline \underset{\sim}{p}^k(t) \end{array}\right]$$

where $\underset{\sim}{y}$ is the reverse system state and $\underset{\sim}{p}$ is the related adjoint vector. The quasilinear equations are

$$\frac{d}{dt}\left[\begin{array}{c} \underset{\sim}{y}^{k+1}(t) \\ \hline \underset{\sim}{p}^{k+1}(t) \end{array}\right] = f(\underset{\sim}{y}^k,\underset{\sim}{p}^k,t) \left[\begin{array}{c} \underset{\sim}{y}^{k+1}-\underset{\sim}{y}^k \\ \hline \underset{\sim}{p}^{k+1}-\underset{\sim}{p}^k \end{array}\right] + \left[\begin{array}{c} -\underset{\sim}{f}(\underset{\sim}{y}^k,t) \\ \hline J_f^T(\underset{\sim}{y}^k,t) \end{array}\right]$$

where F is defined by

$$F(\underset{\sim}{y},p,t) = \left[\begin{array}{c|c} -J_f(\underset{\sim}{y},t) & 0 \\ \hline \frac{\partial}{\partial y}\{J_f^T(\underset{\sim}{y},t)\underset{\sim}{p}\} & J_f^{\;T}(\underset{\sim}{y},t) \end{array}\right]$$

It is convenient to partition the equations further as given by

$$\underset{\sim}{y}(t) = \left[\begin{array}{c} y_1(t) \\ \hline \underset{\sim}{y}_p(t) \end{array}\right] \qquad \underset{\sim}{p}(t) = \left[\begin{array}{c} p_1(t) \\ \hline \underset{\sim}{p}_p(t) \end{array}\right]$$

where the $\underset{\sim}{y}_p$ and $\underset{\sim}{p}_p$ are the parts of $\underset{\sim}{y}$ and $\underset{\sim}{p}$ which remain when y_1 and p_1 are deleted. The solution of the quasilinear equations in terms of this finer partitioning become

$$\left[\begin{array}{c} y_1^{k+1}(T) \\ \hline \underset{\sim}{y}_p^{k+1}(T) \\ \hline p_1^{k+1}(T) \\ \hline \underset{\sim}{p}_p^{k+1}(T) \end{array}\right] = \begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 \\ \phi_3 & \phi_4 & 0 & 0 \\ \phi_5 & \phi_6 & \phi_7 & \phi_8 \\ \phi_9 & \phi_{10} & \phi_{11} & \phi_{12} \end{bmatrix} \left[\begin{array}{c} M \\ \hline \underset{\sim}{y}_p^{k+1}(o) \\ \hline p_1^{k+1}(o) \\ \hline \underset{\sim}{0} \end{array}\right] + \left[\begin{array}{c} w_1 \\ \hline \underset{\sim}{w}_2 \\ \hline w_3 \\ \hline w_4 \end{array}\right]$$

The final states of y and p can be algebraically eliminated from the solution by use of the terminal conditions

$$\left[\begin{array}{c} p_1^{k+1}(T) \\ \hline p_p^{k+1}(T) \end{array}\right] = B\underset{\sim}{v}^{k+1}(T) \triangleq \left[\begin{array}{c|c} b_1 & b_2 \\ \hline b_3 & b_4 \end{array}\right] \left[\begin{array}{c} y_1^{k+1}(T) \\ \hline y_p^{k+1}(T) \end{array}\right]$$

to obtain

$$
\left[\begin{array}{c|c}
-\phi_7 & b_1\phi_2+b_2\phi_4-\phi_6 \\
\hline
-\phi_{10} & b_3\phi_2+b_4\phi_4-\phi_9
\end{array}\right]
\left[\begin{array}{c}
p_1^{k+1}(o) \\
\hline
\underset{\sim}{y}_p^{k+1}(o)
\end{array}\right]
=
\left[\begin{array}{c}
Mb_1\phi_1+b_1w_1+b_2\phi_3M+b_2\underset{\sim}{w}_2-\phi_5M-w_3 \\
\hline
b_3\phi_1M+b_3w_1+b_4\phi_3M+b_4\underset{\sim}{w}_2-\phi_8M-\underset{\sim}{w}_4
\end{array}\right]
$$

Substitution of the computed transition matrix and particular solution into the equation above yields a set of linear algebraic equations with constant coefficients involving the unknown initial conditions. The solution of these algebraic equations produces the initial conditions for the following nonlinear trajectory. These initial conditions are unchanged between successive iterations when the algorithm converges.

If more complicated RC boundaries are specified, then it may not be possible to enforce the boundary conditions on the quasilinear equations by simply solving linear algebraic equations. For example, if the RC boundary is a hyperconical surface, then it is necessary to solve a generalized eigenvalue problem at each iteration of the quasilinearization in order to enforce the boundary conditions.

4.7 Conclusions

Two algorithms are presented above for solving the excursion stability analysis problem. One of these methods, the random search, is based on a reformulation to a parameter optimization problem. Alternatively, the stability problem may be reformulated as a variational problem, and the evolving boundary value problem may be solved by quasilinearization. The random search program is discussed in section 4.3. The parameter usually has a dimension of n - 1 for an n^{th} order system.

The use of quasilinearization to solve the Euler Lagrange equations is discussed in section 4.6. The algorithm requires the repeated initial value integration of a $(4n^2 + 4n)$-order set of differential equations. It is necessary to solve the Euler-Lagrange equations repeatedly for varying durations to solve the stability problem.

The Euler-Lagrange equations cannot be solved by use of the recirculation algorithm. The algorithm is not likely to converge when the duration of the boundary value problem is at all long. This tendency was observed experimentally and also verified, for linear problems, by an error analysis which appears in section 4.5.

A number of simple examples are discussed in the following chapter. These are solved by the random search, and also by quasilinearization. It is anticipated that the random search is more efficient than quasilinearization with regard to computer run time since the order of the equations in the quasilinearization program is so high. The program is also more complicated. It requires about two hundred Fortran statements compared to approximately one hundred for the random search. Neither program requires more than a few thousand words of memory for both program and number storage.

Chapter Five

Examples

5.1 Introduction

The computational algorithms developed in Chapter Four are tested and compared in this chapter by discussing actual computer solutions of several examples using both the random search and the quasilinearization. The programs were carried out on an IBM 7072 digital computer.

Solutions of several simple second order examples are summarized in section 5.2. These examples are taken from Chapter Three. The approximate regions of excursion stability for the examples are obtained in Chapter Three without the use of the computer so that the computer results may be verified. The solution of the same examples using both algorithms produces data for comparing the algorithms on a quantitative basis.

A complete analysis of a third-order system is discussed in section 5.3. The control system under study involves a singly-excited electromechanical transducer. The transducer is highly nonlinear and the describing equations are not amenable to analysis using the extended methods of Popov. Attempts to find a suitable Liapunov function have also been unsuccessful. The analysis begins with a physical description of the system and the manner in which the constraint region and shape function are selected is discussed.

5.2 Simple Examples

Examples 3-3, 3-4 and 3-5 involve linear time-invariant systems. Example 3-3 is analyzed in Chapter Three where it is presented as an example of an approximately maximal system which is not maximally stable. The equations are

$$\frac{d}{dt}\underset{\sim}{x}(t) = A\underset{\sim}{x}(t) = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \underset{\sim}{x}(t)$$

Example 3-4 is a linear system with real eigenvalues which is not approximately maximally stable. The equations for example 3-4 are

$$\frac{d}{dt}\underset{\sim}{x}(t) = \begin{bmatrix} 1 & -1 \\ 6 & -4 \end{bmatrix} \underset{\sim}{x}(t) \quad .$$

Example 3-5 is a linear system with complex eigenvalues which is not approximately maximally stable. The equations are

$$\frac{d}{dt}\underset{\sim}{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \underset{\sim}{x}(t)$$

The constraint region is the same for all three problems, namely

$$RC = \{\underset{\sim}{x} \mid x_1 \leq 1\}$$

The shape function is

$$S(\underset{\sim}{x}) = \underset{\sim}{x}^T\underset{\sim}{x}$$

The random search is conducted by integrating the reverse equations

$$\frac{d}{dt}\,\underset{\sim}{y}(t) = -A\underset{\sim}{y}(t)$$

starting at $y_1 = 1$ and y_2 equal to the trial parameter. The quasilinear equations for the variational method are

$$\frac{d}{dt}\begin{bmatrix} \underset{\sim}{y}^{k+1}(t) \\ \\ \underset{\sim}{p}^{k+1}(t) \end{bmatrix} = \begin{bmatrix} -A & 0 \\ \\ 0 & A^T \end{bmatrix} \begin{bmatrix} \underset{\sim}{y}^{k+1}(t) \\ \\ \underset{\sim}{p}^{k+1}(t) \end{bmatrix}$$

Both the random search and quasilinearization are applied to all of these examples. Both methods yield stability regions which agree well with the stability regions calculated analytically in Chapter Three. The essential results are summarized in Table 1. In each case the run time for the random search is about half that required for solution of the boundary-value problem. The order of the computational task increases approximately according to the square of the system order for quasilinearization, because $(4n^2 + 4n)$ equations are involved. The random search involves only one more parameter for each new state variable, so that the order of the computational task increases approximately linearly with the system order. Table 1 shows that the random search is superior to quasilinearization for second-order systems. The use of variational techniques is, therefore, not recommended until more efficient algorithms are developed for the solution of boundary value problems.

Several other interesting observations are apparent from the table. The quasilinearization converged in one iteration for the time invariant linear problems as predicted in Chapter Four. The random

TABLE 1

COMPARISON OF QUASILINEARIZATION AND RANDOM SEARCH METHODS AS APPLIED TO THE EXAMPLES OF SECTION 5-2

Example Number	3-3	3-4	3-5	3-6
Run Time for Random Search on IBM 7072	1 1/2 minutes	1 1/2	1 1/2	6
Run Time for Quasilinearization on IBM 7072	3 minutes	3	3	11
Average Number of Quasilinear Iterations	1	1	1	3
Number of Random Trials Attempted throughout Program	188	135	244	116
Number of Trials before last Success Occurred	63	36	85	32

search program obtained the critical point in approximately thirty-five trials, but it continued for a hundred or more trials before it reached a program stop. Therefore, the random search spent most of its time verifying rather than seeking the global minimum.

The run times are approximate and include compilation time. The compiler is in Fortran II and is conducted on an IBM 1401. The program is run on an IBM 7072. The duration increment separating members of the sequence of boundary value problems is 0.1 seconds. The random search limited evaluations of the reverse trajectories to a maximum of 6 seconds and the quasilinearization was limited to 2 seconds. Some program constants relevant to the random search, as defined in Chapter Four, are listed below as they are used in the random search program.

Initial Noise Variance	0.01
Maximum Number of Trials	300
Variance Reduction Factor	0.8
Minimum Step Size	0.005
Maximum Number of Successive Failures in Local Mode	10
Portion of Bias Retained at each Success	75%
Maximum Shape Function	10

These selections are made largely on the basis of the experience of G. A. Bekey (1964).

Table 1 also summarizes the results of application of both methods to example 3-6, which is nonlinear and contains an unstable limit cycle. The reverse equations for the random search are taken directly from Chapter Three. The quasilinear equations are obtained from these reverse equations by application of eqn.(4-5). The reverse equations are

$$\frac{d}{dt}\,\underset{\sim}{y}(t) = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & -1 \end{bmatrix} \underset{\sim}{y}(t)+[4y_1^2(t)+y_2^2(t)]^{-\frac{1}{2}}\;\underset{\sim}{y}(t)$$

where the superscript k has been deleted. The Jacobian matrix is

$$J_f(\underset{\sim}{y}) = \begin{bmatrix} 1 & \frac{1}{2} \\ -2 & 1 \end{bmatrix} + [4y_1^2(t)+y_2^2(t)]^{-\frac{3}{2}} \begin{bmatrix} -y_2^2(t) & 4y_1(t)y_2(t) \\ y_1(t)y_2(t) & -4y_1^2(t) \end{bmatrix}$$

The adjoint equations are

$$\frac{d}{dt}\,\underset{\sim}{p}(t) = J_f(y)\underset{\sim}{p}(t)$$

The elements of the matrix F in the quasilinear equations are listed below, after deleting time arguments

$$\mathrm{RAD} = \sqrt{4y_1^2 + y_2^2}$$

$$F_{11} = -1 + y_2^2\,\mathrm{RAD}^{-3}$$

$$F_{12} = -\frac{1}{2} - 4y_1y_2\,\mathrm{RAD}^{-3}$$

$$F_{13} = 0 = F_{14}$$

$$F_{21} = 2 - y_1y_2\,\mathrm{RAD}^{-3}$$

$$F_{22} = -1 + 4y_1^2\,\mathrm{RAD}^{-3}$$

$$F_{23} = 0 = F_{24}$$

$$F_{31} = y_2p_2\,\mathrm{RAD}^{-3}-12y_1[y_1y_2p_2-y_2^2p_1]\,\mathrm{RAD}^{-5}$$

$$F_{32} = [-2p_1y_2+y_1p_2]\ \overline{RAD}^3 - 12y_1[4y_1y_2p_1 - 4y_1^2p_2]\ \overline{RAD}^5$$

$$F_{33} = 1 - y_2^2\ \ \overline{RAD}^3$$

$$F_{34} = -2 + y_1y_2\ \overline{RAD}^3$$

$$F_{41} = [4p_1y_2 - 8p_2y_1]\ RAD^3 - 3y_2(y_1y_2p_2 - y_2^2p_1)\ \overline{RAD}^5$$

$$F_{42} = 4p_1y_1\ \overline{RAD}^3 - 3y_2(4y_1y_2p_1 - 4y_1^2p_2)\ \overline{RAD}^5$$

$$F_{43} = \frac{1}{2} + 4y_1y_2\ \overline{RAD}^3$$

$$F_{44} = 1 - 4y_1^2\ \overline{RAD}^3$$

The linear terms in the quasilinear equations are

$$[F] \quad \begin{bmatrix} y^{k+1}(t) \\ \hline p^{k+1}(t) \end{bmatrix}$$

The equivalent forcing functions in the quasilinear equations are

$$u_1(t) = -3y_1y_2^2\ \overline{RAD}^3 + y_1\ RAD$$

$$u_2(t) = -3y_1^2y_2\ \overline{RAD}^3 + y_2\ RAD$$

$$u_3(t) = [-2y_1y_2p_2 - 2y_2^2p_1]\ \overline{RAD}^3 - 36y_1^2\ [y_1y_2p_2 - y_2^2p_1]\ \overline{RAD}^5$$

$$u_4(t) = -4p_1(p_1y_2+p_2y_1)\ \overline{RAD}^3 + 3y_2p_1[y_1y_2p_2 - y_2^2p_1]\ \overline{RAD}^5$$

The equations above illustrate the fact that the quasilinear equations become very cumbersome when the system equations are nonlinear.

5.3 Analysis of a Control System

The system analyzed in this section involves a singly-excited electro-magnetic transducer, a third-order system with several product-type nonlinearities. It makes an excellent example for the demonstration of excursion stability analysis for a number of reasons. The constraints which must be satisfied are visible, as the mechanical shaft of the transducer has a limited range of translation. The nonlinearities are severe enough so that they cannot be ignored. Since there are product-type nonlinearities, the Popov methods are not applicable. The equations are rather cumbersome so that it is very difficult to find a Liapunov function for the system equations, if it is possible to find such a function at all, even though the system is only third-order.

The closed loop system consists of the transducer itself, a mechanical load composed of a linear spring and linear friction, and a controller which selects the e.m.f. applied to the transducer coil on the basis of measurements of the coil current and the shaft position. The control law is selected on the following basis. First of all, a nonlinear term with a large coefficient is cancelled from the equation by way of the control function. The remainder of the control law involves a linear combination of the measurable states with the coefficients selected by synthesis of the linearized system. The analysis is also repeated with the nonlinear feedback term removed for comparison. The transducer itself is cylindrical in shape. A section of the transducer is shown in Fig. 27.

The system is being analyzed for its quality as a regulator. The desired shaft position is halfway between the mechanical stops. The stops are placed to prevent the shaft from slamming into the pole piece and to prevent the inside end of the shaft from moving past the inside edge of the shaft bearing. These limits on the shaft position define the constraint region.

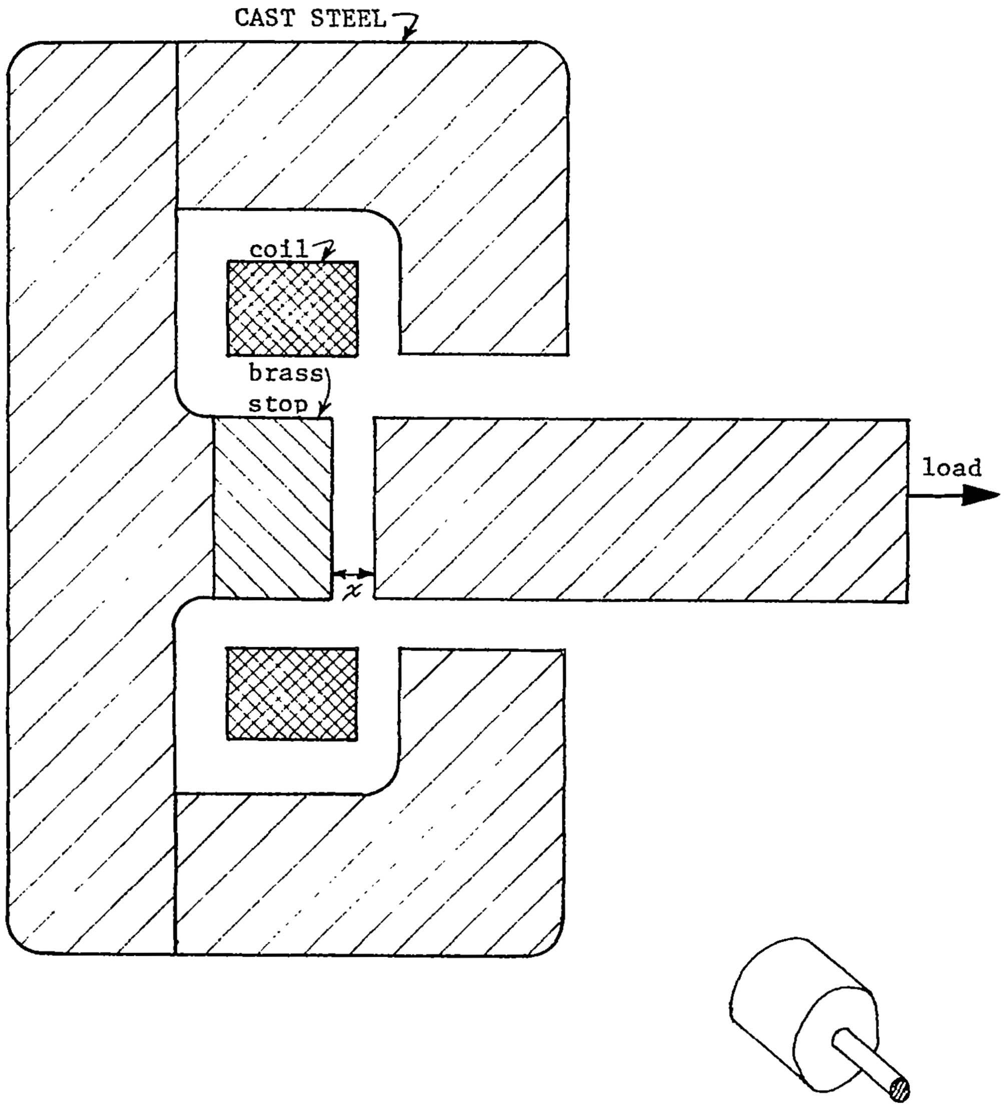

Figure 27. Singly-Excited Transducer

The describing equations for the transducer are obtained by White and Woodson (1959) using Lagrange's equations. The system is also discussed by Schultz and Melsa (1967). The equations are

$$M \frac{d^2}{dt^2} x(t) + \alpha \frac{d}{dt} x(t) - K_s(D-x(t)) - f_e = 0 \tag{5-1}$$

$$e(t) = Ri(t) + \frac{\partial\lambda(i,x)}{\partial i} \frac{d}{dt} i(t) + \frac{\partial\lambda(i,x)}{\partial x} \frac{d}{dt} x(t) \tag{5-2}$$

$$f_e = \int_o^i \frac{\partial\lambda(\gamma,x)}{\partial x} d\gamma \tag{5-3}$$

where x and i are the length of the variable portion of the air gap and the current in the coil respectively. M is the mass of the shaft. K_s and alpha are the proportionality constants for the linear load. The fixed portion of the air gap is denoted by d. D is the projected value of x for which the load spring is relaxed. The e.m.f. imposed on the coil is e and the coil resistance is R. Lambda represents the flux linkages in the coil. If the iron has a high permeability and saturation level, then the flux linkages in the coil are given in MKS units by

$$\lambda = \frac{0.081N^2 i(t)}{2+100x(t)} \tag{5-4}$$

Suppose M is one kilogram, K_s is one newton per meter, alpha is one newton per meter per second and D and d are both two centimeters. The state equations for the open loop system are then obtained by substituting eqn. (5-4) into eqn. (5-3) to obtain

$$\frac{d}{dt} x_1(t) = x_2(t)$$

$$\frac{d}{dt} x_2(t) = -x_1(t) - x_2(t)+2 - \frac{0.0405N^2x_3^2(t)}{(2+x_1(t))^2}$$

$$\frac{d}{dt} x_3(t) = (x_1(t)+2)/(0.081N^2)[100e-Rx_3(t)]+x_2(t)x_3(t)/2+x_1(t)$$

where x_1 is equal to x in centimeters, x_2 is the first time derivative of x in centimeters per second and x_3 is the coil current in ten-milliamp units. The turn count N is expressed in thousand-turn units.

If the maximum allowable variable portion of the air gap is two centimeters, then the median position is when x_1 equals one. At equilibrium all time derivatives must be zero so that

$$0 = x_2$$

$$0 = -1 - x_2 + 2 - \frac{0.0405N^2x_3^2}{(2+1)^2}$$

$$0 = \frac{3}{0.081N^2} \quad (100E - Rx_3) + \frac{x_2x_3}{3}$$

where E is the equilibrium value of e. These equations show that the equilibrium values of the state variables are

$$x_1 = 1 \qquad x_2 = 0 \qquad x_3 = \frac{15}{N}$$

and also that the equilibrium value of e must be

$$E = 0.15\,\frac{R}{N}$$

The following transformation shifts the equilibrium point to the origin

$$x_1^1 = x_1 - 1$$

$$x_2^1 = x_2$$

$$x_3^1 = x_3 - \frac{15}{N}$$

$$u = e - E$$

If this transformation is performed, and the primes are dropped for convenience, the equations become

$$\frac{d}{dt}\,x(t) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & -1 & -0.133N \\ 0 & \frac{5}{N} & -\frac{37R}{N^2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ g_2 \\ g_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}$$

where

$$g_2 = \frac{-3x_1^2 + 0.8Nx_1x_3 - 0.04N^2x_3^2 - \frac{2}{3}x_1^3}{(x_1 + 3)^2}$$

$$g_3 = \frac{x_2x_3 - \frac{5}{N}x_1x_3}{x_1 + 3} - 12.34\,\frac{R}{N^2}\,x_1x_3$$

The closed loop system is formed by feeding x_1 and x_3 back to u according to the control law

$$u(x) = 7.15x_1 + 374x_3 + 12.34\,\frac{R}{N^2}\,x_1x_3$$

The linear feedback coefficients have been selected so that the linearized system for R = 10 and N = 1 has two critically damped poles and one real pole well into the left half plane. The characteristic equation for the linearized system is

$$S^3 + (1 + 37R)S^2 + (1 + 37R)S + (12.3R - 0.951) = 0$$

The nonlinear term in the control law is inserted to diminish the effect of the nonlinearity in the x_3 equation by cancelling the analogous term in g_3. The analysis is repeated without this nonlinear feedback to demonstrate its effectiveness.

The shape function is determined by assuming the following relationship between the probable disturbances in all three states. The disturbances in x_2 are likely to be approximately 50% larger than those in x_1. Moreover, it is likely that disturbances in the first two state variables accompany one another and that they occur with the same polarity. If something pushes momentarily on the shaft, then it causes a translation of the shaft from equilibrium and also causes the shaft to acquire velocity in the same sense from equilibrium. The disturbances in the electrical circuit are expected to be, on the average, only about one-third of the probable x_2 disturbances. These weightings are considered in the establishment of the following shape function

$$S(\underset{\sim}{x}) = 2x_1^2 + (x_1-x_2)^2 + 10x_3^2 = \underset{\sim}{x}^T \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix} \underset{\sim}{x}$$

The constraint region is

$$RC = \{\underset{\sim}{x} \mid |x_1| \leq 1\}$$

so two separate runs are made on the problem. First the constraint is

$$x_1 \leq 1$$

is imposed and then the problem is rerun with the constraint

$$x_1 \geq -1$$

The smaller approximate region of the two solutions obtained above is a region from which neither constraint is violated.

The results are obtained with two minutes of running time on each constraint. The limiting constraint is

$$x_1 \geq -1$$

which is the prevention of hitting the pole piece. The value of the shape function on the boundary of the approximate stability region is 0.640. The critical point is at

$$x_1 = -0.565 \qquad x_2 = -0.551 \qquad x_3 = 0.001$$

The critical point for reaching the other constraint is

$$x_1 = 0.971 \qquad x_2 = 1.003 \qquad x_3 = 0.002$$

at which the shape function value is 1.886.

If the nonlinear term in the control law is removed, then the same constraint limits the stability region and the critical point is at

$$x_1 = -0.382 \qquad x_2 = -C.570 \qquad x_3 = 0.001$$

which has a shape function value of 0.474. The desirable effect of the nonlinear feedback is now evident from the significant decrease in the excursion stability region in its absence. The evaluation of the region in the modified case again took two minutes on each constraint.

The solution of this example by quasilinearization is not reported. The reverse equations are

$$\frac{d}{dt} y_1 = -y_2$$

$$\frac{d}{dt} y_2 = \frac{1}{3} y_1 + y_2 + 0.133 y_3 - \frac{0.8 y_1 y_3 - 0.04 y_3^2 - \frac{2}{3} y_1^3 - 3 y_1^2}{(y_1 + 3)^2}$$

$$\frac{d}{dt} y_3 = 7.15y_1 + 5y_2 + 3y_3 - \frac{y_2y_3 - 5y_1y_3}{y_1 + 3}$$

and the Jacobian matrix has the following elements

$$J_{11} = J_{13} = 0 \qquad\qquad J_{12} = 1$$

$$J_{21} = -\frac{1}{3} - (0.8y_1y_3 - 0.04y_3^2 - \frac{2}{3}y_1^3 - 3y_1^2)\frac{2}{(y_1+3)^3} - \frac{0.8y_3-2y_1^2-6y_1}{(y_1 + 3)^2}$$

$$J_{22} = -1$$

$$J_{23} = -0.133 + \frac{0.8y_1 - 0.08y_3}{(y_1 + 3)^2}$$

$$J_{31} = -7.15 - \frac{5y_3}{(y_1 + 3)} + \frac{5y_1y_3}{(y_1 + 3)^2}$$

$$J_{32} = -5 + \frac{y_3}{(y_1 + 3)}$$

$$J_{33} = -3 + \frac{y_2 - 5y_1}{y_1 + 3}$$

The quasilinear equations are grotesquely elaborate. There is little hope that the method would converge for long. For example, the fourth row first column element of the F matrix is

$$F_{41} = p_2 \frac{\partial}{\partial y_1} J_{21} + p_3 \frac{\partial}{\partial y_1} J_{31}$$

This situation emphasizes another advantage of the random search. The random search requires only that the system equations be programmed for solution. In contrast, quasilinearization requires the preparation of very cumbersome equations if the nonlinearities are nontrivial and the order is anything past two.

5.4 Conclusion

The examples discussed in this chapter illustrate that the random serach is easy to use and reasonably efficient. The quasilinearization is considerably less efficient and more difficult to use. The random search is a more powerful tool. The computation time definitely increases with the system order, but a more dramatic cause of high computation time is the presence of an unstable limit cycle. The reason for this effect is that the reverse of an asymptotically stable trajectory quickly diverges from the origin so that the shape function increases rapidly and the minimum shape function value along the trajectory is quickly evaluated. However, an unstable limit cycle becomes a stable limit cycle for the reverse equations so that many reverse trajectories linger in the vicinity of the origin for a long time before the program is confident that the trajectory won't approach the origin.

The availability of fast digital computers would make it possible to consider far larger examples on a digital program. However, to consider very high-order examples, it is necessary to reduce the time required for integration of the reverse equations. A hybrid digital analog facility is ideal for this purpose.

Chapter Six

Summary and Conclusion

6.1 Introduction

This chapter contains a summary of the entire work and lists some first thoughts related to possible extensions of this research. The summary is contained in section 6.2. The extensions of the work fall into three categories. The analysis might be broadened to consider a more general environment or a more general class of systems. For example, distributed parameter systems have not been studied. The analysis of a system under the influence of continuous disturbing forces is suggested by the variational formulation of the excursion stability problem. This possibility is considered further in section 6.3. Another category of extensions arise from interest in the related synthesis problem. A number of approaches are possible. One suggestion relating to synthesis procedures is discussed in section 6.4. Finally, the variational algorithm might be made more efficient so that the analysis can be executed on a totally digital program. The possibility of directly determining the locus of terminal points of the optimal trajectories is considered in section 6.5.

6.2 Summary

The basic objective of the study is to revise stability analysis techniques so that they are more useful and more easily conducted.

Stability analysis is broken down into two questions in Chapter One. How should stability be defined? How can a given system be tested for stability? In Chapters Two and Three it is suggested that a system should be considered stable if it satisfies the most rudimentary operational requirements. These qualifications are adequately expressed in terms of state-space constraints for regulator systems. The system is considered stable if the disturbances which are likely to occur leave the system in such a state that the constraints are not violated.

The analysis problem requires finding the states from which the free system does not violate the specified constraints. This problem is solved by reformulation to a previously studied problem and invoking existing techniques. Simplifications or methods which inherently or effectively limit the order of the systems which are amenable to analysis are avoided. It is suggested that the problem be reformulated as a parameter optimization problem. The parameter optimization may be performed by a random search. Third-order problems may be studied on a digital computer. Higher-order problems are most easily handled if a coupled analog computer is available to integrate the differential equations. Another possibility is to reformulate as a variational problem. The problem is well defined and leads to a set of two-point boundary value problems which must be solved to find the stability region. The solutions to the boundary value problems are not easy to obtain. The variational formulation is, however, an interesting one and it suggests a modified analysis.

The boundary value problems referred to above are attacked with the recirculation algorithm and quasilinearization. The former is shown to be unsuitable for the problem of interest. Quasilinearization yields results at high computational expense. There are many factors contributing to this high cost. The order of the boundary value problem is twice that of the system under study and a whole sequence of boundary value problems must be solved. Quasilinearization requires the repeated initial value integration of differential equations many times the order of the system studied. This order magnification rate increases rapidly as the system order increases.

The emphatic increase in computational efficiency which the random search enjoys over quasilinearization, as applied to the stability problem, is demonstrated by several examples presented in Chapter Five. However, before either program can be initiated a great deal of information about the system under study must be accumulated. The selection of RC requires a careful study of the basic limitations of the system being controlled. The selection of the shape function requires some knowledge of the disturbances to which the system is likely to be subjected. However, if the system limitations are to be meaningfully considered in the analysis, then there is no alternative to obtaining a detailed description of the system.

If the system limitations are important enough for consideration in analysis, then they are certainly no less important in system synthesis. System design techniques based on excursion stability would be valuable to the engineer. Expanded applicability and improved efficiency of the methods developed in this work are also interesting problems for future study. These extensions are considered briefly in the remainder of the chapter.

6.3 Stability in the Presence of Amplitude Bounded Disturbances

The variational problem associated with excursion stability analysis is solved in terms of a boundary value problem by application of the maximum principle in Chapter Four. The equations are

$$\frac{d}{dt}\,\underset{\sim}{y}(t) = -\,\underset{\sim}{f}(\underset{\sim}{y}(t),t)$$

$$\frac{d}{dt}\,\underset{\sim}{p}(t) = J_f(\underset{\sim}{y}(t),t)\underset{\sim}{p}(t)$$

and the boundary values are

$$G(\underset{\sim}{y}(o)) = 0$$

$$\nabla_y G(\underset{\sim}{y}(o)) = \underset{\sim}{p}(o)$$

$$\nabla_y S(\underset{\sim}{y}(T)) = \underset{\sim}{p}(T)$$

There are states from which unstable trajectories may emanate when a certain disturbance is present even if those states are stable in the absence of continuous disturbances. The disturbances, therefore, reduce the size of the approximate stability region. Let $\underset{\sim}{u}$ be the vector disturbance which reduces the stability region the most. Suppose the disturbance vector must satisfy the following amplitude constraint

$$|\,u_i\,| \le u_i\max$$

The worst case trajectory solves

$$\frac{d}{dt}\,\underset{\sim}{y}(t) = -\underset{\sim}{f}(\underset{\sim}{y}(t),t) + \underset{\sim}{u}(t)$$

$$\frac{d}{dt}\, \underset{\sim}{p}(t) = J_f(\underset{\sim}{y}(t),t)\underset{\sim}{p}(t)$$

subject to the same boundary conditions stated above. The disturbance, $\underset{\sim}{u}$, is some function of the adjoint variables $\underset{\sim}{p}$. In the absence of singular solutions

$$u_i = u_{i\max} \text{ Sign } (p_i)$$

The constraint on the forcing function usually causes the worst case disturbance to be an on-off function.

The amplitude constraint seems to be a natural one, but other constraints on the disturbance function might be of interest. Some possibilities are constraints on the time derivative of the disturbance or constraints on its frequency content.

6.4 Terminal Point Loci

There is a possibility that the sequence of boundary value problems evolving from the variational approach to the stability problem might succumb to more efficient algorithms than those which are available for solving each problem in the sequence individually. The useful information in the solution of any one of the problems is exhibited in the terminal value of the reverse variables. If the terminal points of the solutions of the boundary value problems were known, then the whole problem of finding RS(S) would effectively be solved. It is possible to obtain a set of differential equations which contain the locus of terminal points as a part of their solution. Unfortunately, the equations also involve several other variables for which the initial conditions are not apparent. If someone should uncover a method of estimating the initial value of these related variables, then it would be possible to obtain the whole locus of terminal points from a single initial value problem.

The locus equations are heuristically obtained by applying the technique of invariant imbedding to the sequence of boundary value problems. The development presented below follows the plausibility argument for nonlinear estimation filter equations by Detchmendy and Sridhar (1965).

The boundary value problems involve the reverse variables $\underset{\sim}{y}(t)$ and the adjoint variables $\underset{\sim}{p}(t)$. The adjoint variables are replaced by a variable, given the symbol $\underset{\sim}{z}(t)$, as indicated below

$$\underset{\sim}{z}(t) \stackrel{\Delta}{=} \underset{\sim}{p}(t) - B\underset{\sim}{y}(t)$$

where B is a symmetric nxn matrix which occurs in the definition of the quadratic shape function

$$S(\underset{\sim}{x}) = \underset{\sim}{x}^T B \underset{\sim}{x}$$

The restated boundary value problem involves $\underset{\sim}{y}(t)$ and $\underset{\sim}{z}(t)$ in the equations

$$\frac{d}{dt}\underset{\sim}{y}(t) = -\underset{\sim}{f}(\underset{\sim}{y}(t),t)$$

$$\frac{d}{dt}\underset{\sim}{z}(t) = J_f(\underset{\sim}{y}(t),t)[\underset{\sim}{z}(t)+B\underset{\sim}{y}(t)] + B\underset{\sim}{f}(\underset{\sim}{y}(t),t) \stackrel{\Delta}{=} \underset{\sim}{g}(\underset{\sim}{y},\underset{\sim}{z},t)$$

The boundary conditions become $G(\underset{\sim}{y}(o)) = 0$

$$\nabla_y G(\underset{\sim}{y}(o)) = \underset{\sim}{z}(o) + B\underset{\sim}{y}(o)$$

$$\underset{\sim}{z}(T) = \underset{\sim}{0}$$

Let $\underset{\sim}{r}(\underset{\sim}{w},T)$ represent the terminal value of $\underset{\sim}{y}$ when the final condition above is replaced by the more general condition given by

$$\underset{\sim}{z}(T) = \underset{\sim}{w}$$

If the relevant differential equations are integrated past the indicated terminal condition for a very short time interval ΔT, then the resulting states of $\underset{\sim}{y}$ and $\underset{\sim}{z}$ are given to within a first order approximation by

$$\Delta\underset{\sim}{y}(T) = \Delta\underset{\sim}{r}(\underset{\sim}{w},T) = -\underset{\sim}{f}(\underset{\sim}{r}(w,T),T)\Delta T \tag{6-1}$$

$$\Delta\underset{\sim}{z} = \underset{\sim}{g}(\underset{\sim}{r}(\underset{\sim}{w},T),\underset{\sim}{w}(T),T)\Delta T \tag{6-2}$$

A Taylor Series expansion of the function $\underset{\sim}{r}(\underset{\sim}{w},T)$ truncated after the first-order terms gives

$$\Delta\underset{\sim}{r}(\underset{\sim}{w},T) = \frac{\partial\underset{\sim}{r}(\underset{\sim}{w},T)}{\partial\underset{\sim}{w}}\,\Delta\underset{\sim}{w} + \frac{\partial\underset{\sim}{r}(\underset{\sim}{w},T)}{\partial T}\,\Delta T \tag{6-3}$$

Substituting eqn. (6-2) into eqn. (6-3) and equating to eqn. (6-1) gives

$$-\underset{\sim}{f}(\underset{\sim}{r}(\underset{\sim}{w},T),T)\Delta T = \frac{\partial\underset{\sim}{r}(\underset{\sim}{w},T)}{\partial\underset{\sim}{w}}\,\underset{\sim}{g}(\underset{\sim}{r}(\underset{\sim}{w},T),\underset{\sim}{w},T)\Delta T + \frac{\partial\underset{\sim}{r}(\underset{\sim}{w},T)}{\partial T}\,\Delta T$$

Divide both sides of the above equation by ΔT and let ΔT go to zero. The higher-order terms which were ignored above vanish, and the following partial differential equation remains

$$-f(\underset{\sim}{r}(\underset{\sim}{w},T),T) = \frac{\partial \underset{\sim}{r}(\underset{\sim}{w},T)}{\partial \underset{\sim}{w}} \underset{\sim}{g}(\underset{\sim}{r}(\underset{\sim}{w},T),\underset{\sim}{w},T) + \frac{\partial \underset{\sim}{r}(\underset{\sim}{w},T)}{\partial T} \qquad (6\text{-}4)$$

The $\underset{\sim}{r}(\underset{\sim}{w},T)$, $\underset{\sim}{f}(\underset{\sim}{r}(\underset{\sim}{w},T),T)$ and $\underset{\sim}{g}(\underset{\sim}{r}(\underset{\sim}{w},T),\underset{\sim}{w},T)$ are expanded about $\underset{\sim}{w}$ equal zero, the result is

$$\underset{\sim}{r}(\underset{\sim}{w},T) = \underset{\sim}{r}(\underset{\sim}{o},T) + \left.\frac{\underset{\sim}{r}(\underset{\sim}{w},T)}{\partial \underset{\sim}{w}}\right|_{\underset{\sim}{w}=0} \underset{\sim}{w} = \underset{\sim}{R}(T) + Q(T)\underset{\sim}{w}$$

$$\underset{\sim}{f}(\underset{\sim}{r}(\underset{\sim}{w},T),T) = \underset{\sim}{f}(\underset{\sim}{R}(T),T) + \frac{\partial \underset{\sim}{f}(\underset{\sim}{R}(T),T)}{\partial \underset{\sim}{R}} Q(T)\underset{\sim}{w} \quad \underset{\sim}{g}(\underset{\sim}{r}(\underset{\sim}{w},T),\underset{\sim}{w},T)$$

$$= \underset{\sim}{g}(\underset{\sim}{R}(T),\underset{\sim}{0},T) + \frac{\partial \underset{\sim}{g}(\underset{\sim}{R}(T),\underset{\sim}{0},T)}{\partial R} \quad Q(T)\underset{\sim}{w}$$

These expansions may be substituted into the partial differential equation to obtain

$$-\underset{\sim}{f}(\underset{\sim}{R}(T),T) - \frac{\partial \underset{\sim}{f}(\underset{\sim}{R}(T),T)}{\partial \underset{\sim}{R}} Q(T)\underset{\sim}{w} = Q(T)\underset{\sim}{g}(\underset{\sim}{R}(T),\underset{\sim}{0},T)$$

$$+ \quad Q(T) \frac{\partial \underset{\sim}{g}(\underset{\sim}{R}(T),0,T)}{\partial \underset{\sim}{R}} Q(T)\underset{\sim}{w} + \frac{d\underset{\sim}{R}(T)}{dT} + \frac{dQ(T)}{dT} \quad \underset{\sim}{w}$$

Equating like powers of $\underset{\sim}{w}$ yields

$$-\underset{\sim}{f}(\underset{\sim}{R}(T),T) = Q(T)\underset{\sim}{y}(\underset{\sim}{R}(T),\underset{\sim}{0},T) + \frac{d\underset{\sim}{R}(T)}{dT} \qquad (6\text{-}5)$$

$$- \frac{-\partial \underset{\sim}{f}(\underset{\sim}{R}(T),T)}{\partial \underset{\sim}{R}} \quad Q(T) = Q(T) \frac{\partial \underset{\sim}{g}(\underset{\sim}{R}(T),0,T)}{\partial \underset{\sim}{R}} \quad Q(T) + \frac{dQ(T)}{dT} \qquad (6\text{-}6)$$

In terms of the original equations, eqn. (6-5) and eqn. (6-6) reduce to

$$\frac{d\underset{\sim}{R}(T)}{dT} = -\underset{\sim}{f}(\underset{\sim}{R}(T),T)-Q(T)\{J_f(R(T),T)B\underset{\sim}{R}(T)+B\underset{\sim}{f}(\underset{\sim}{R}(T),T)\} \qquad (6\text{-}7)$$

where

$$\frac{dQ(T)}{dT} = -\frac{\partial \underset{\sim}{f}(\underset{\sim}{R}(T),T)}{\partial \underset{\sim}{R}} Q(T) - Q(T)h(\underset{\sim}{R}(T),T)Q(T) \qquad (6\text{-}8)$$

$$h(\underset{\sim}{R}(T),T) = \frac{\partial}{\partial R}\Bigg|_{\underset{\sim}{w}=0} \{J_f(\underset{\sim}{R}(T),T)[\underset{\sim}{w}+B\underset{\sim}{R}(T)] + B\underset{\sim}{f}(\underset{\sim}{R}(T),T)\}$$

The square matrix Q(T) satisfies eqn. (6-8). The simultaneous solution of eqn. (6-7) produces the desired terminal point locus. The initial conditions on $\underset{\sim}{R}(T)$ are the solutions of a zero-interval boundary value problem. That is, $\underset{\sim}{R}(0)$ is available from the solution of an algebraic equation. However, Q(0) is not readily available. If a number of solutions of the original sequence of problems are available, then Q(0) may be computed by solving an(n + 1)-point boundary value problem involving eqn. (6-7) and eqn. (6-8). This multipoint boundary value problem involving such a high-order system is more of a computational task then the solution of the original sequence of two-point problems.

There may be an easier way to obtain the initial value of Q(T). If such an evaluation can be made, then the technique outlined above can be used to solve the stability problem quite efficiently as well as solving many other problems which can be reformulated as a sequence of boundary value problems.

6.5 Synthesis

The ultimate objective of control engineering is to select a control policy for a given system on the basis of a meaningful goodness criterion. If the extent of the region of excursion stability is suitable for system evaluation, then it should also be a useful criterion for system design.

An obvious extension of the stability analysis techniques to the synthesis problem would be to optimize the selection of design parameters on the basis of the size of the approximate region of excursion stability. A straightforward way of making such a selection is to use the random search technique outlined in Chapter Four. The extent of RS(S) is graded in terms of design parameters, $\underset{\sim}{v}$ by evaluating the shape function at the critical piont associated with a particular $\underset{\sim}{v}$ such as

$$S(\underset{\sim}{x}_{critical}, \underset{\sim}{v})$$

The choice of $\underset{\sim}{v}$ which maximizes the function above is an optimum design.

A more general design problem requires the selection of the structure of the controller. It would be interesting to determine when and how a system could be made maximally stable. It is sometimes necessary to compromise in the selection of RC if maximal stability is to be achievable. For example, consider a second-order system described in phase variables with an inequality constraint on x_1. If the control policy affects the time derivative of x_2 only, then it is not possible to prevent trajectories from escaping from RC in the upper half of the phase plane where x_1 is always increasing.

The chances of forcing maximal stability appear to be enhanced if the constraint region is hyperconical. Suppose a system is described by

$$\frac{d}{dt}\underset{\sim}{x}(t) = \underset{\sim}{f}(\underset{\sim}{x}(t),t) + \underset{\sim}{b}u(\underset{\sim}{x})$$

where the constraint region is given by

$$\underset{\sim}{x}^T(t)E\underset{\sim}{x}(t) \leq 1 \qquad \text{(E symmetric)}$$

and the control law u(x) is under the discretion of the designer. The direction of penetration of the RC boundary by system trajectories is given by the sign of the state function D(x) defined below

$$D(\underset{\sim}{x}) = \underset{\sim}{x}^T(t)E\ \underset{\sim}{f}(\underset{\sim}{x}(t),t)$$

The control law desired can be selected as follows. At points along the RC boundary where D is negative, let u be zero. At points where D is zero on the RC boundary but

$$\underset{\sim}{x}^T(t)E\ \underset{\sim}{b} \neq 0$$

then let u be given by

$$u = -(1.1)\ \frac{\underset{\sim}{x}^T(t)E\underset{\sim}{f}(\underset{\sim}{x}(t),t)}{\underset{\sim}{x}^T(t)E\underset{\sim}{b}}$$

If there are points on the RC boundary where

$$\underset{\sim}{x}^T(t)E\underset{\sim}{f}(\underset{\sim}{x}(t),t) \geq 0 \quad \text{and } \underset{\sim}{x}^T(t)E\underset{\sim}{b} = 0$$

then there is no choice of bounded u which will force maximal stability. The selection of the control law for points inside RC can be made so that the behavior in RC is acceptable.

The discussion above does not detail a procedure for establishing a control law, but it does outline some of the possible methods which might be studied. The selection of u on the boundary is not unique. Perhaps in some cases it is possible to devise a control law which is independent of some of the state variables thus allowing some transducers or estimation schemes to be eliminated.

It would be beneficial to study a simple design problem on the basis of excursion stability and to correlate the size of RS(S) with other characteristics of the design such as the eccentricity of the trajectories or the settling time to the origin from certain disturbed states.

6.6 Conclusion

The novelty of this study among other works in stability analysis is a consequence of the awareness of two premises. One premise is simply that the differential equations are not adequate to describe a control system for engineering purposes. The physical limitations of the system and the nature of the disturbing influences should also be represented in some quantitative fashion. The other premise is that problems which are complex enough to justify use of the computer should be formulated at the outset in terms of representations and methods which are natural to the computer. Analytical methods for finding V-functions are difficult to modify for numerical computation.

The degree to which these premises have ultimately guided this work cannot be immediately determined, but each premise is believed to restate sound engineering principles which are worthy of consideration by other students of stability analysis.

Appendix

An Application of Excursion Stability to
Nuclear Reactor Dyanmics

A.1 Introduction

The stability analysis of nuclear reactors typically involves the study of a highly nonlinear system of describing equations. These stability studies are also very important since safety is of prime concern in reactor design. The definition of excursion stability is very natural to the basic engineering problem of reactor stability so it is not surprising that the techniques of excursion stability should be applied first in this area.

The convergence of these two studies was initiated during a brief visit by the author to Los Alamos Scientific Laboratory. The analog-digital hybrid facility was programmed for the random search technique by a joint effort with Dr. Hugh Murray and Charles Millich, both of L. A. S. L. These engineers provided excellence in nuclear reactor dynamics and computer programming, respectively.

The primary goal of the effort at Los Alamos was to write and "de-bug" the random search program for use on the available computational facilities. This program could be used for excursion stability analysis and also for other parameter optimization problems. The next objective was to begin the analysis of a point model nuclear reactor with coupled two temperature region reactivity feedback. This reactor is under study by Dr. Richard Brehm and Dr. Lynn Weaver of the Nuclear Engineering Department at the University of Arizona.

Both of these objectives were investigated. However, the usable random search program was not refined and the study of the reactor stability was performed for only one set of values of the two design parameters. A summary of the effort directed toward both objectives appears below followed by a discussion of the results.

A.2 Random Search Program for L. A. S. L.

The hybrid computer facility used at L. A. S. L. was comprised of an EAI-231R analog computer and a DDP-116 digital computer with a 16K memory and optional high speed arithmetic. These are linked by forty D to A and forty A to D conversion channels. The system has bilateral mode control capability.

The executive routine and logical decision functions of the random search program are performed on the DDP-116 and the simulation of the reverse system is delegated to the EAI-231R. The logical functions require much less time than the system simulation so the two computers are operated in alternate steps. However, the DDP-116 does periodically sample the simulation state and compute the performance index during the simulation so that the cost function can be extracted and analog overloads avoided.

The program is basically the same as that outlined in Section 4.3. The gross functions are diagrammed in Fig. A-1.

The start-up routine simply involves feeding in the initial guess and setting all the search control parameters. It is assumed that the analyst is present at the time the program is executed so that these numbers can be "typed-in." The advantage of typing in the data is that poorly selected search control parameters may be improved by restarting the program utilizing the previous "best value" without having to punch cards or tape.

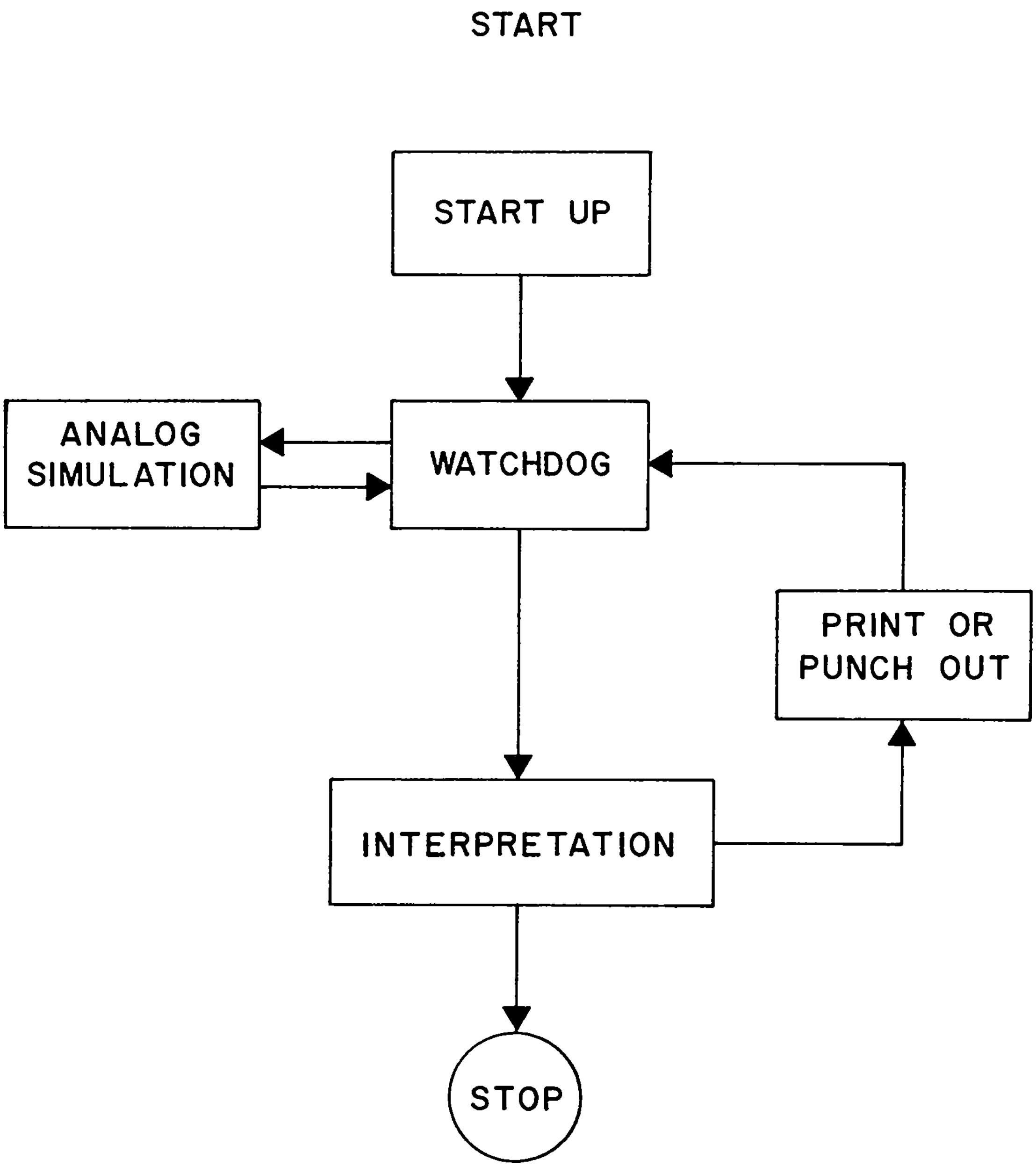

Figure A-1. Block Diagram of Information Flow

The watchdog routine sets the initial conditions on the analog computer and monitors the simulation recording the minimum value of the performance index and terminating the simulations of poor trials quickly.

The interpretation routine classifies the previous trial and computes initial conditions for a new trial. The details of this routine are identical to the analogous steps in Fig. 24.

The digital program is the same for all parameter optimization problems. An object tape for the program retained by N-4 Group at Los Alamos and a copy is held by the Nuclear Engineering Department at the University of Arizona.

There are three steps in the preparation of a problem for use of the program above.

Step One: Diagram the Analog Simulation of the Reverse System.

Step Two: Time and Magnitude Scale the Simulation.

Step Three: Select the first guess, the coefficients of the quadratic performance index and the search control parameters.

Step One is clear and Step Three is discussed in the text. Step Two must be done very carefully. Since the EAI 231R is a relatively slow machine, the problem was time scaled so that a maximum of one second of real time represented the longest simulated time of interest. Another second of real time is necessary for recovery at each iteration. Magnitude scaling presented some serious problems and emphasized the need for high precision analog facilities. The problem must be scaled so that a region somewhat larger than the region of maximal stability may be penetrated and yet the critical point might be well inside the region of maximal stability. Therefore, the range of all state variables must be determined and an analog computer with good performance when signals diminish to as little as one percent of full scale is necessary. The former requirement might mean that the whole problem may have to be rescaled and rerun after the first run. The latter requirement limits the speed at which the analog simulation can be run.

A.3 Reactor Analysis

The reactor described by the following equations is of interest to the Nuclear Engineering faculty at Arizona.

$$(0.01-\rho)\dot{n} = (\dot{\rho} + 0.1\rho)(n + n_o)$$

$$\rho = \alpha_1 T_1 + \alpha_2 T_2$$

$$\dot{T}_1 = 100n - (0.46T_1 + 3.3\cdot 10^{-4} T_1^{\,2} + 10^{-7} T_1^{\,3} + 1.2\cdot 10^{-11} T_1^{\,4})$$

$$\dot{T}_2 = 0.23T_1 + 1.65\cdot 10^{-4} T_1^{\,2} + 5\cdot 10^{-6} T_1^{\,3} + 6\cdot 10^{-10} T_1^{\,4} - 0.2T_2$$

with constraint

$$-500 \le T_1 \le 1500$$

where n = neutron density

ρ = reactivity

T_1 = temperature in region one

T_2 = temperature in region two

Define H by

$$H = 4.6\cdot 10^{-2}T_1 + 3.3\cdot 10^{-5}T_1^{\ 2} + 10^{-8}T_1^{\ 3} + 1.2\cdot 10^{-12}T_1^{\ 4}$$

The following magnitude scaling was used:

T_1, T_2, H	by	1700
n	by	5
α_1, α_2	by	10^{-7}
ρ	by	10^{-3}

The magnitude scaled equations are

$$\dot{T}_1 = 29.41n - 10H$$

$$\dot{T}_2 = 5H - 0.2T_2$$

$$\dot{n} = \frac{0.017\alpha_1T_1 - 0.323\alpha_2T_2 + 0.5\alpha_1n - 1.7\alpha_1H + 8.5\alpha_2H}{10 - 0.17\alpha_1T_1 - 0.17\alpha_2T_2}(n+0.2)$$

$$\rho = 0.17\alpha_1T_1 + 0.17\alpha_2T_2$$

The equilibrium point has been shifted to the origin above. The ultimate goals of the analysis are to determine the size of the excursion stability region for the shape function below

$$S(\underline{x}) = (\frac{n}{5})^2 + (\frac{T_1}{1700})^2 + (\frac{T_2}{1700})^2$$

in the region of the design parameter space near the point

$$\alpha_1 = 0.3\cdot 10^{-5}$$

$$\alpha_2 = -0.32\cdot 10^{-5}$$

However, time limitations allowed only the determination of the critical point based on the nominal parameter values and the positive constraint on T_1.

A.4 Results

The problem described in Section A.2 was approached with the program described in A.1. The program was terminated after about 350 iterations. The critical point indicated on Step 245 is as follows (in MACHINE units)

$$S(\underline{x}_{crit}) = 0.0297$$

$$\text{Critical Point} \quad \begin{cases} T_1 = -0.028 \\ T_2 = 0.007 \\ n = 0.344 \end{cases}$$

Dr. Brehm's associates indicated that these values appear very reasonable except for T_2. Since the critical T_2 is less than one percent of full scale, it is likely that there is an error. The problem should perhaps be redone with T_2 rescaled.

There was a problem with excessive run time for the experiment. The computation time for a complete iteration averaged a little better than two seconds. However, the 350 iterations took several hours of running time. The excessive time was lost punching the output on paper tape. A four-word heading and nine numbers were punched out at each iteration regardless of whether the trial was a success or failure. This is clearly far more than the required output because diagnostic information was desired. However, if such complete observance is desired, high-speed printout hardware is necessary. The "answer" can be obtained from the final success which can be retained and printed at the end of the entire run.

There are several definite, if not positive, conclusions which can be drawn from the study. First of all, it is feasible to determine excursion stability on a hybrid computer. Also, the method seems to be usable for the study of such systems as reactors which have dominating nonlinearities.

Finally, the magnitude scaling of the problem requires great care and the analog computer precision is the factor which places the limits on what problems can be solved on a particular installation.

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